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Finite Element Programming for Incompressible Fluids

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Chapter 1

Functional Preliminaries

In this chapter, we will investigate the concepts used throughout the weighted residual and finite element methods. These concepts can be thought of as originating from an area of mathematics called functional analysis. This chapter is intended to familiarize the reader with some concepts of functional analysis. Although, the chapter is not a complete review of functional analysis, pertinent material is covered to allow for a complete understanding of the material presented in this text. We will begin by recalling a subject most engineering students have seen in their junior or senior years, Fourier series representation of a function. We shall investigate a property of these Fourier functions (sine and cosine) used to construct the series, namely orthogonality. We shall use another property of these Fourier functions to develop a concept of functional spaces. In each functional space, we shall define the concept of basis functions and spanning functions. We shall study the concepts of linear independence and completeness to force a set of functions to span a functional space. Using these concepts we shall investigate two important function sets used in finite element methods, Legendre and Tchebychev polynomials. We shall use these polynomial functions to expand any function as a series. We shall develop a transformation matrix between each of these polynomial sets and a monomial set. Next we review a orthonormalization method over 100 years old yet extremely powerful, Gram-Schmidt orthonormalization. Using this method, we may take any basis functions and reorient these functions into a set of orthonormal basis functions. Finally, one of the most powerful mathematical theorems with respect to the weighted residual and finite element method is presented, the *first fundamental lemma of variational calculus*. Studying this theorem, in the context of the next chapter, we see why the weighted residual and finite element methods work so well.

1.1 Orthogonal Functions

Two functions are orthogonal if they obey the following relationship:

$$\int_{x_1}^{x_2} f(x)\, g(x)\, dx = 0 \tag{1.1}$$

These two functions, $f(x)$ and $g(x)$ are orthogonal in the range $x_1 \leq x \leq x_2$. This property is used in Fourier analysis to obtain a Fourier series representation of a periodic function. We

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will use Fourier series representation of functions to investigate several properties of orthogonal functions. Given any function $f(x)$, we may represent this function as a Fourier series over the range $-\pi \leq x \leq \pi$ as,

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx) \tag{1.2}$$

where a_0, a_n and b_n are unknown constants. If we are able to find these constants, the function $f(x)$ may be represented as a Fourier series. Orthogonality is the property that makes finding these constants possible. If we look at a set of integral relationships, we will make use of the following orthogonal relationships:

$$\begin{aligned}
\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx &= \pi\delta_{nm} \\
\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx &= \pi\delta_{nm} \\
\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,dx &= 0 \\
\int_{-\pi}^{\pi} \cos(mx)\,dx &= 0 \\
\int_{-\pi}^{\pi} \sin(mx)\,dx &= 0
\end{aligned} \tag{1.3}$$

From these relationships we notice several orthogonality conditions, shown in the table below.

$g(x)$	$f(x)$	Orthogonality Condition	Orthogonality Domain
$\sin(mx)$	$\sin(nx)$	$m \neq n$	$-\pi \leq x \leq \pi$
$\cos(mx)$	$\cos(nx)$	$m \neq n$	$-\pi \leq x \leq \pi$
$\sin(mx)$	$\cos(nx)$	Always	$-\pi \leq x \leq \pi$
1	$\cos(mx)$	Always	$-\pi \leq x \leq \pi$
1	$\sin(mx)$	Always	$-\pi \leq x \leq \pi$

As we shall see shortly, these orthogonality conditions are used to obtain a Fourier series representation of a function over the range $-\pi \leq x \leq \pi$. Any function, whether periodic or not, may be represented as Fourier series over the range $-\pi \leq x \leq \pi$. Although not crucial to our development, this range limitation may be removed by a simple coordinate transformation:

$$x = c_1 + c_2 y \tag{1.4}$$

This transformation may be used to map any domain back to the range $-\pi \leq x \leq \pi$ where the Fourier series may be expressed. For example, if constant c_1 is equal to π and constant c_2 is 2 then our mapping would be,

$$x = \pi + 2y \tag{1.5}$$

As a result, the range of each domain would be, $-\pi \leq x \leq \pi$ and $-\pi \leq y \leq 0$.

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Even Function (a_m's)

To find our constants a_m we multiply equation 1.2 by $\cos(mx)$ and integrate over the range $-\pi \leq x \leq \pi$.

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx = &\int_{-\pi}^{\pi} \cos(mx)\,\frac{1}{2}a_0\,dx \\ &+ \int_{-\pi}^{\pi} \cos(mx)\sum_{n=1}^{\infty} a_n\cos(nx)\,dx \\ &+ \int_{-\pi}^{\pi} \cos(mx)\sum_{n=1}^{\infty} b_n\sin(nx)\,dx\end{aligned} \tag{1.6}$$

Rearranging integrations and summations:

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx = &\frac{a_0}{2}\underbrace{\int_{-\pi}^{\pi}\cos(mx)\,dx}_{=0\ \text{(Orthogonal)}} \\ &+ \sum_{n=1}^{\infty} a_n\int_{-\pi}^{\pi}\cos(mx)\cos(nx)\,dx \\ &+ \sum_{n=1}^{\infty} b_n\underbrace{\int_{-\pi}^{\pi}\cos(mx)\sin(nx)\,dx}_{=0\ \text{(Orthogonal)}}\end{aligned} \tag{1.7}$$

Removing the orthogonal terms (zero term):

$$\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx = \sum_{n=1}^{\infty} a_n\underbrace{\int_{-\pi}^{\pi}\cos(mx)\cos(nx)\,dx}_{=\pi\delta_{nm}} \tag{1.8}$$

Orthogonality again helped by replacing the integral relationship on the right hand side by the term $\pi\delta_{mn}$. This term may be understood by observing the definition of the delta, δ_{nm}.

$$\delta_{nm} = \begin{cases} 0 & \text{iff } n \neq m \\ 1 & \text{iff } n = m \end{cases} \tag{1.9}$$

Therefore, our equation becomes,

$$\boxed{a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx} \tag{1.10}$$

Note,

$$\sum_{n=1}^{\infty} a_n\pi\delta_{nm} = a_1\pi\delta_{1m} + a_2\pi\delta_{2m} + a_3\pi\delta_{3m} + \cdots + a_\infty\pi\delta_{\infty m} \tag{1.11}$$

If, for example, m were equal to 3 this equation would become,

$$\sum_{n=1}^{\infty} a_n\pi\delta_{n3} = a_1\pi\underbrace{\delta_{13}}_{=0} + a_2\pi\underbrace{\delta_{23}}_{=0} + a_3\pi\underbrace{\delta_{33}}_{=1} + \cdots + a_\infty\pi\underbrace{\delta_{\infty 3}}_{=0} = a_3\pi \tag{1.12}$$

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Odd Function (b_m's)

To find our constants b_m we multiply equation 1.2 by $\sin(mx)$ and integrate over the range $-\pi \leq x \leq \pi$.

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx = \frac{1}{2}a_0 \underbrace{\int_{-\pi}^{\pi}\sin(mx)\,dx}_{=0\text{ (Orthogonal)}} \\ + \sum_{n=1}^{\infty} a_n \underbrace{\int_{-\pi}^{\pi}\sin(mx)\cos(nx)\,dx}_{=0\text{ (Orthogonal)}} \\ + \sum_{n=1}^{\infty} b_n \underbrace{\int_{-\pi}^{\pi}\sin(mx)\sin(nx)\,dx}_{=\pi\delta_{nm}}\end{aligned} \tag{1.13}$$

Removing orthogonal term gives,

$$\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx = \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi}\sin(mx)\sin(nx)\,dx = \pi\delta_{nm} \tag{1.14}$$

Dividing by π and making use of the definition of δ_{mn} our constant b_m may be written as,

$$\boxed{b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx} \tag{1.15}$$

Constant Function, (a_0)

To find our constant a_0 we integrate equation 1.4 over the range $-\pi \leq x \leq \pi$.

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2}a_0 \int_{-\pi}^{\pi} dx \\ + \sum_{n=1}^{\infty} a_n \underbrace{\int_{-\pi}^{\pi}\cos(nx)\,dx}_{=0\text{ (Orthogonal)}} \\ + \sum_{n=1}^{\infty} b_n \underbrace{\int_{-\pi}^{\pi}\sin(nx)\,dx}_{=0\text{ (Orthogonal)}}\end{aligned} \tag{1.16}$$

Or

$$\boxed{a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx} \tag{1.17}$$

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1.1.1 Upcoming Definitions and Theorems

We shall use the Fourier series to help in the explanation of several definitions. These definitions include the following,

- Functional spaces
- Spanning a functional space
- Complete set of functions
- Linearly independent set of functions
- Basis functions

In addition, we will explain, through use of an orthogonal series, the most powerful theorem used in weighted residual as well as the finite element, the *first fundamental lemma of variational calculus*.

Functional Spaces

Some finite element textbooks tend toward the mathematical side. In those texts, often terms such as Banach, Sobolev and Hilbert spaces are discussed, see [4]. These spaces are functional spaces which may be classified in some particular way. Exactly what are functional spaces? To answer this question, we will use our Fourier series to define our own functional spaces. Using the Fourier series, we can define three distinct functional spaces.

Functional Space I

We can define functional space I as all functions that are constant in the range, $-\pi \leq x \leq \pi$.

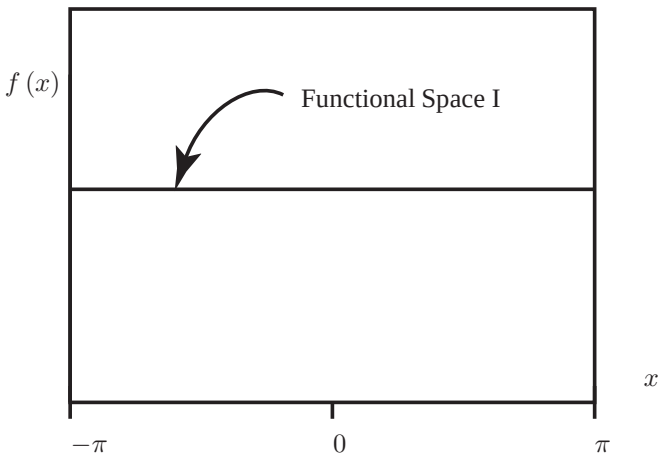


Figure 1.1: Space I

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This space may be written mathematically as,

$$\boxed{f(x) = C_I} \tag{1.18}$$

Interestingly enough the Fourier series has one term which can handle all functions in this space, shown below with the underbrace label of "Space I".

$$f(x) = \underbrace{\underbrace{\underbrace{\frac{1}{2}a_0}_{\text{Space I}} + \sum_{n=1}^{\infty} a_n \cos(nx)}_{\text{Space II}} + \underbrace{\sum_{n=1}^{\infty} b_n \sin(nx)}_{\text{Space III}}}_{\text{Space IV}} \tag{1.19}$$

From the Fourier series equation the first term is all that is required to express any function in Space I. As a result, this first term *spans* space I. In other word all constant functions may be written as,

$$f(x) = \frac{1}{2}a_0 = C_I \tag{1.20}$$

The unknown in this equation a_0 is simply twice the value of our constant function, C_I. This is the simplest example of a *spanning* set of functions in that there is only one function, $\frac{1}{2}$ and one constant, a_0. If things seem a bit foggy at this point, read on, space II and III may clarify the concepts functional spaces and spanning sets of functions.

Functional Space II

Functional space II, is defined as all even functions in the range, $-\pi \leq x \leq \pi$. An even function is defined as,

$$\boxed{f(x) = f(-x)} \tag{1.21}$$

Examples of these functions include, x^2, x^4, $\cos(x)$, $\cos(2x)$. These functions are shown in the figure below.

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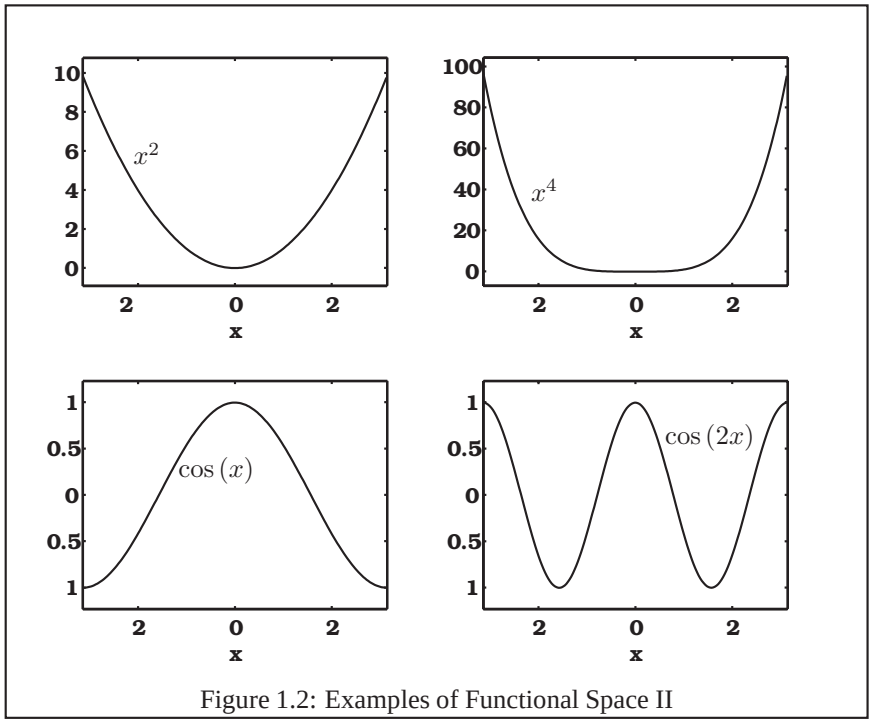


Figure 1.2: Examples of Functional Space II

All functions in this space may be expressed in the form,

$$f_{II}(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) \tag{1.22}$$

We see this section of the Fourier series in equation 1.19. Notice the under-braces with space labels on equation 1.19. We notice immediately that functional space I is a subset of functional space II. In other words, a constant function obeys the space I and II defining relationship, equations 1.18 and 1.21. However, not all even functions are constant, for example $f_{II}(x) = x^2$ is an even function and is not a constant. Therefore it belongs to functional space II and not to functional space I. From this we learn that one functional space may be a subset of another.

From observing equation 1.22 we determine a spanning set of functions. These spanning functions are,

$$\phi_n(x) = \left\{ \begin{array}{c} \phi_0(x) \\ \phi_1(x) \\ \phi_2(x) \\ \phi_3(x) \\ \phi_4(x) \\ \vdots \end{array} \right\} = \left\{ \begin{array}{c} \frac{1}{2} \\ \cos(x) \\ \cos(2x) \\ \cos(3x) \\ \cos(4x) \\ \vdots \end{array} \right\} \tag{1.23}$$

We use this new notation to rewrite equation 1.22 as,

$$f_{II}(x) = \sum_{n=0}^{\infty} a_n \phi_n(x) \tag{1.24}$$

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This is a convenient time to introduce a new notation:

$$\mathcal{B}(\phi_i, \phi_j) = \mathcal{B}(\phi_j, \phi_i) = \int_{-\pi}^{\pi} \phi_i(x)\,\phi_j(x)\,dx \tag{1.25}$$

This notation $\mathcal{B}(\phi_i, \phi_j)$ is commonly called a bilinear form. Interestingly enough, *for our Fourier series*, this bilinear form and orthogonality are related by equation 1.26. This orthogonality relationship will not always hold for the bilinear form. However, by definition if the equations below are satisfied we have a set of orthogonal basis functions.

$$\begin{aligned} \mathcal{B}(\phi_i, \phi_j) &= 0 \quad \text{if} \quad i \neq j \\ \mathcal{B}(\phi_i, \phi_j) &\neq 0 \quad \text{if} \quad i = j \end{aligned} \tag{1.26}$$

We will keep this in mind and reuse this again in the next functional space.

Definitions

At this point it seems appropriate to place a definition on a spanning function set.

> A set of functions $\phi_i(x)$ *span a space* if all functions in that space may be represented as a linear combination of the functions $\phi_i(x)$, see equation 1.24. This set of functions is often called *basis* functions.

It is worth noting here that all spanning sets of functions are by definition complete.

> A set of functions $\phi_i(x)$ is complete if all terms from lowest to highest order are included.

It is also worth noting that a complete set of functions may not be a spanning set. However a spanning set of functions is by definition a complete set. These definitions may be clarify by looking at possible cases from functional space II. The set of functions shown in equation 1.27 is not a complete set. The reason for this incompleteness is the omission of the term $\cos(3x)$.

$$\phi_n(x) = \left\{ \begin{array}{c} \phi_0(x) \\ \phi_1(x) \\ \phi_2(x) \\ \phi_3(x) \\ \phi_4(x) \end{array} \right\} = \left\{ \begin{array}{c} \frac{1}{2} \\ \cos(x) \\ \cos(2x) \\ \cos(4x) \\ \cos(5x) \end{array} \right\} \tag{1.27}$$

The set shown in equation 1.28 is a complete set. Every term from the lowest order to the highest order is included in the set.

$$\phi_n(x) = \left\{ \begin{array}{c} \phi_0(x) \\ \phi_1(x) \\ \phi_2(x) \\ \phi_3(x) \\ \phi_4(x) \\ \phi_5(x) \\ \phi_6(x) \end{array} \right\} = \left\{ \begin{array}{c} \frac{1}{2} \\ \cos(x) \\ \cos(2x) \\ \cos(3x) \\ \cos(4x) \\ \cos(5x) \\ \cos(6x) \end{array} \right\} \tag{1.28}$$

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Neither set of functions, equation 1.27, nor equation 1.28 spans functional space II. To find out why we select any function in functional space II, say $\cos(3x)$. This function is in functional space II because $\cos(3x) = \cos(-3x)$, see equation 1.21. If we try to express this function as a linear combination of basis functions, equation 1.27 we have a problem.

$$\begin{aligned} \cos(3x) &= \sum_{i=0}^{4} a_i \phi_i(x) \\ \cos(3x) &\neq \frac{a_0}{2} + a_1 \cos(x) + a_2 \cos(2x) + a_3 \cos(4x) + a_4 \cos(5x) \end{aligned} \tag{1.29}$$

It is not possible to satisfy this equation regardless of the what value the constants a_0 through a_4 are assigned. The reason for this inequality is the set of functions in equation 1.27 are incomplete. If all terms from the constant $\frac{1}{2}$ all the way up to $\cos(5x)$ were included this equation could be satisfied.

If we now consider the function $\cos(12x)$ we have a similar problem. The set of basis functions in equation 1.27 may not be used to represent this function.

$$\begin{aligned} \cos(12x) &= \sum_{i=0}^{4} a_i \phi_i(x) \\ \cos(12x) &\neq \frac{a_0}{2} + a_1 \cos(x) + a_2 \cos(2x) + a_3 \cos(4x) + a_4 \cos(5x) \end{aligned} \tag{1.30}$$

Although the result is the same as with the previous case, equation 1.29, the reasons are different. Even if we reinserted the missing term in this sequence, $\cos(3x)$ into equation 1.27, the inequality would still exist. The reason for this inequality is the set of functions is not large enough to capture this higher order term. However, if we had extended the functional set, equation 1.28, to infinity it would span the functional space II and therefore, *any* even function could be expressed as a linear combination of these spanning functions, including our function $\cos(12x)$.

$$\cos(12x) = \sum_{n=0}^{\infty} a_n \phi_n(x) \tag{1.31}$$

Which would give a solution, $a_0 = a_1 = \cdots = a_{11} = a_{13} = \cdots = a_\infty = 0$ and $a_{12} = 1$.

One final definition should be covered at this point and will be used extensively throughout this text is that of linear dependency and independency.

A set of functions $\phi_i(x)$ is linearly independent if no one term may be written as a linear combination of all the remaining terms. Mathematically a set of linearly independent functions obey,

$$\phi_k(x) \neq \sum_{i=0}^{k-1} a_i \phi_i(x) + \sum_{i=k+1}^{N} a_i \phi_i(x) \tag{1.32}$$

for all k from 0 to ∞.

This definition is best described with several examples. We begin by looking at the functional set in equation 1.27 and asking the equation: is every term in this functional set linearly

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independent? To answer this question we write our functional set as shown in equation 1.33.

$$\begin{aligned}
\frac{1}{2} \neq & a_1 \cos(x) + a_2 \cos(2x) + a_3 \cos(4x) + a_4 \cos(5x) \\
\cos(x) \neq & \frac{a_0}{2} + a_2 \cos(2x) + a_3 \cos(4x) + a_4 \cos(5x) \\
\cos(2x) \neq & \frac{a_0}{2} + a_1 \cos(x) + a_3 \cos(4x) + a_4 \cos(5x) \\
\cos(4x) \neq & \frac{a_0}{2} + a_1 \cos(x) + a_2 \cos(2x) + a_4 \cos(5x) \\
\cos(5x) \neq & \frac{a_0}{2} + a_1 \cos(x) + a_2 \cos(2x) + a_3 \cos(4x)
\end{aligned} \tag{1.33}$$

There is no combination of constants, a_0 through a_4 that are possible to make *any* of these equations equal. Since no term in the functional set may be written as a linear combination of any of the other terms in the set, these set of functions are linearly independent. It should be noted the functional set we used, equation 1.27, is incomplete. As a result, we may have an incomplete functional set which is linearly independent. Likewise, we may have a complete set which is linearly dependent.

If we looked at a functional set shown in equation 1.34 we notice there is a linear dependence in this set.

$$\phi_n(x) = \left\{ \begin{array}{c} \phi_0(x) \\ \phi_1(x) \\ \phi_2(x) \\ \phi_3(x) \\ \phi_4(x) \end{array} \right\} = \left\{ \begin{array}{c} \frac{1}{2} \\ \cos(x) \\ \cos(2x) \\ 5 + \cos(x) \\ \cos(3x) \end{array} \right\} \tag{1.34}$$

If we express every term as a linear combination of all the other terms, yielding 5 equations, we notice there is one equation which is an equality:

$$\begin{aligned}
\frac{1}{2} \neq & a_1 \cos(x) + a_2 \cos(2x) + a_3 [5 + \cos(x)] + a_4 \cos(3x) \\
\cos(x) \neq & \frac{a_0}{2} a_2 \cos(2x) + a_3 [5 + \cos(x)] + a_4 \cos(3x) \\
\cos(2x) \neq & \frac{a_0}{2} + a_1 \cos(x) + a_3 [5 + \cos(x)] + a_4 \cos(3x) \\
[5 + \cos(x)] = & \frac{a_0}{2} + a_1 \cos(x) + a_2 \cos(2x) + a_4 \cos(3x) \\
\cos(3x) \neq & \frac{a_0}{2} + a_1 \cos(x) + a_2 \cos(2x) + a_3 [5 + \cos(x)]
\end{aligned} \tag{1.35}$$

In fact, if we observe the fourth equation in this set we may verify the solution, $a_0 = 10$, $a_1 = 1$, $a_2 = 0$ nd $a_4 = 0$. Therefore, the term $5 + \cos(x)$ is a linear combination of the terms, $\frac{1}{2}$ and $\cos(x)$. It is worth noting that this *linearly dependent* functional set is a complete set with all terms up to $\cos(3x)$ included.

In review the following may be said about a functional set.

- A set of functions, $\phi_i(x)$, spans a space if all functions which exist in this space may be written as a linear combination of these functions, $\phi_i(x)$, See equation 1.24.

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- A spanning functional set is both linearly independent and complete. These functions are commonly called *basis* functions, much the same way as $\vec{i}$, $\vec{j}$, $\vec{k}$ are *basis* vectors in x-y-z vector space.

- Any functional set that is complete up to order n, $\cos(nx)$ in our example, may be used to express functions of equal or lower order. In effect, this functional set spans a space which is a subset of the larger space. For example, if we define a new space by "functions that are even and of order equal to or lower than $\cos(nx)$". This space may be spanned by a linearly independent and complete functional set, up to, and including order n.

- A complete functional set is not necessarily linearly independent.

- A linearly independent functional set is not necessarily complete.

- Any functional set which spans a space must be a linearly independent set of functions. Otherwise, it will not be possible to uniquely define all the constants in an expansion, equation 1.24.

- Any functional set which spans a space must be a complete. Otherwise, it will not be possible to obtain a solution to equation 1.24 if in fact the function on the left hand side is the function omitted, $\phi_k(x)$, in the right hand side expansion, see equation 1.29.

With these definitions we continue on, looking at our next functional spaces.

Functional Space III

Functional space III, is defined as all odd functions in the range, $-\pi \leq x \leq \pi$. An odd function is defined as,

$$\boxed{f(x) = -f(-x)} \tag{1.36}$$

Examples of this type of functions include, x, x^3, $\sin(x)$, $\sin(2x)$, and are shown in the figure below.

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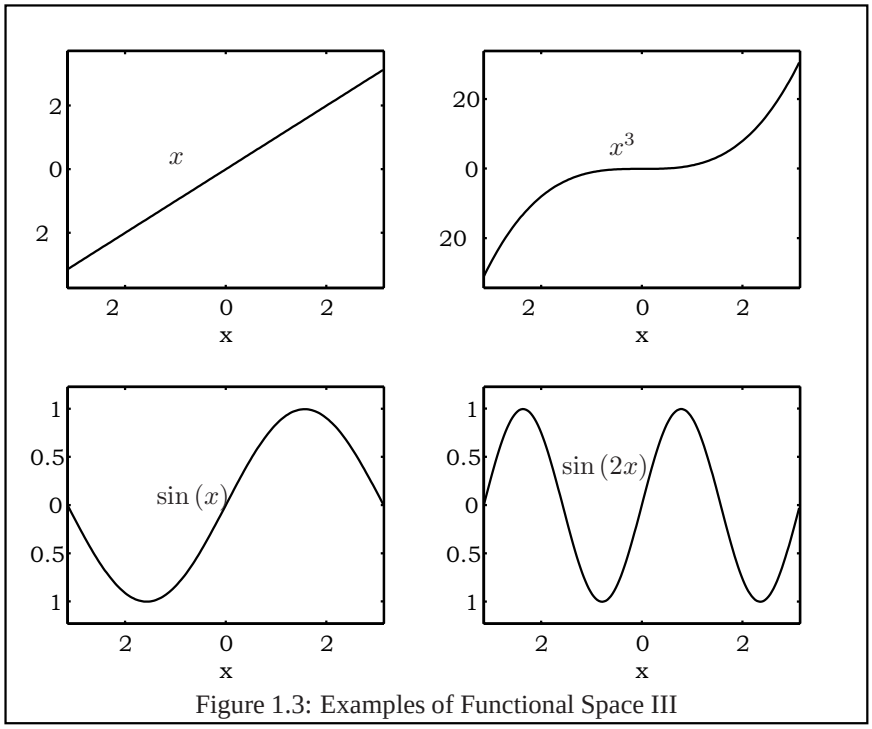


Figure 1.3: Examples of Functional Space III

All functions in this space may be expressed in the form,

$$f_{III}(x) = \sum_{n=1}^{\infty} b_n \sin(nx) \tag{1.37}$$

Notice the under-braced labels on equation 1.19. We may rewrite our equation 1.37 as,

$$f_{III}(x) = \sum_{n=1}^{\infty} a_n \varphi_n(x) \tag{1.38}$$

Where,

$$\varphi_n(x) = \left\{ \begin{array}{c} \varphi_1(x) \\ \varphi_2(x) \\ \varphi_3(x) \\ \varphi_4(x) \\ \vdots \end{array} \right\} = \left\{ \begin{array}{c} \sin(x) \\ \sin(2x) \\ \sin(3x) \\ \sin(4x) \\ \vdots \end{array} \right\} \tag{1.39}$$

Again, our bilinear form may be written as,

$$\mathcal{B}(\varphi_i, \varphi_j) = \mathcal{B}(\varphi_j, \varphi_i) = \int_{-\pi}^{\pi} \varphi_i(x)\, \varphi_j(x)\, dx \tag{1.40}$$

For our functions, Fourier series, the bilinear form and orthogonality is related by:

$$\begin{array}{ll} \mathcal{B}(\varphi_i, \varphi_j) = 0 & \text{if } i \neq j \\ \mathcal{B}(\varphi_i, \varphi_j) \neq 0 & \text{if } i = j \end{array} \tag{1.41}$$

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As stated before, the Fourier series is an orthogonal set of equations. As a result the bilinear from gives zeros for all terms which have different indices, equation 1.41. It is well worth noting, not all basis function need be orthogonal, and most times basis functions are not orthogonal.

Functional Space IV

Functional space IV, is defined as all functions in the range, $-\pi \leq x \leq \pi$. These functions may be written in terms of a Fourier series, equation 1.2. We may write this set of functions in a more compact form as,

$$f(x) = \sum_{i=0}^{\infty} c_i \psi_i(x) \tag{1.42}$$

Where the spanning function set, basis functions, are,

$$\psi_n(x) = \left\{ \begin{array}{c} \psi_0(x) \\ \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \\ \psi_5(x) \\ \psi_6(x) \\ \vdots \end{array} \right\} = \left\{ \begin{array}{c} \frac{1}{2} \\ \sin(x) \\ \cos(x) \\ \sin(2x) \\ \cos(2x) \\ \sin(3x) \\ \cos(3x) \\ \vdots \end{array} \right\} \tag{1.43}$$

After a careful inspection of equations 1.3 and the subsequent table, we determine this set of functions is orthogonal.

$$\begin{array}{ll} \mathcal{B}(\psi_i, \psi_j) = 0 & \text{if } i \neq j \\ \mathcal{B}(\psi_i, \psi_j) \neq 0 & \text{if } i = j \end{array} \tag{1.44}$$

Where, the bilinear relationship is,

$$\mathcal{B}(\psi_i, \psi_j) = \mathcal{B}(\psi_j, \psi_i) = \int_{-\pi}^{\pi} \psi_i(x) \psi_j(x) \, dx \tag{1.45}$$

Example

In this section we consider an example of a function expressed as a Fourier series. Consider the function shown in figure 1.4. This ramp function can be written in equation form for three distinct zones.

$$\begin{array}{ll} f(x) = +1 & -\pi \leq x \leq -2 \\ f(x) = -\frac{x}{2} & -2 \leq x \leq 2 \\ f(x) = -1 & 2 \leq x \leq \pi \end{array} \tag{1.46}$$

We used the orthogonality conditions, equation 1.3, to generate equations 1.10, 1.15 and 1.17. To find the Fourier series representation of our function, equation 1.46 we simply use these equations. We begin by finding our constant term a_0 from equation 1.17.

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$$\begin{aligned} a_0 =& \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ =& \frac{1}{\pi}\int_{-\pi}^{2} dx + \frac{1}{\pi}\int_{-2}^{2} -\frac{x}{2}dx + \frac{1}{\pi}\int_{2}^{\pi} dx \\ =& 0 \end{aligned} \tag{1.47}$$

Using equation 1.10 we can determine the contribution from the cosine in the series.

$$\begin{aligned} a_m =& \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx \\ =& \frac{1}{\pi}\int_{-\pi}^{2} \cos(mx)\,dx \\ &+ \frac{1}{\pi}\int_{-2}^{2} -\frac{x}{2}\cos(mx)\,dx \\ &- \frac{1}{\pi}\int_{2}^{\pi} \cos(mx)\,dx \\ =& 0 \end{aligned} \tag{1.48}$$

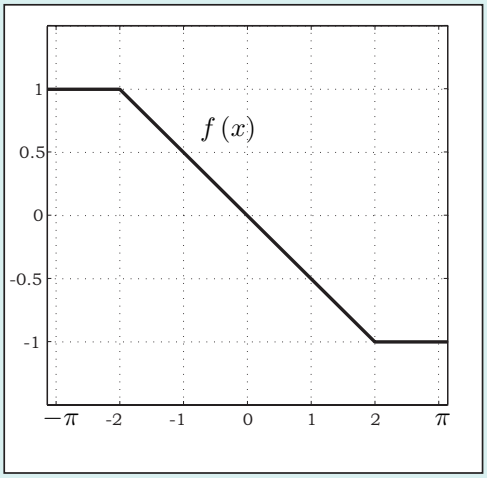


Figure 1.4: Ramp Function

This result, equations 1.47 and 1.48, make perfect sense due to the nature of our function. From figure 1.4, we notice our function is in functional space III. I.E. our function is odd and therefore obeys equation 1.36. As a result, there is no cosine contribution to our series. The entire series contains only contributions from the sin functions, equation 1.15.

$$\begin{aligned} b_m =& \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx \\ =& \frac{1}{\pi}\int_{-\pi}^{2} \sin(mx)\,dx \\ &+ \frac{1}{\pi}\int_{-2}^{2} -\frac{x}{2}\sin(mx)\,dx \\ &- \frac{1}{\pi}\int_{2}^{\pi} \sin(mx)\,dx \\ =& \frac{2\,m\cos(\pi\,m) - \sin(2\,m)}{m^2\pi} \end{aligned} \tag{1.49}$$

Our Fourier series representation of the ramp function may be written as,

$$f(x) = \sum_{m=1}^{\infty}\left[\frac{2\,m\cos(\pi\,m) - \sin(2\,m)}{m^2\pi}\right]\sin(mx) \tag{1.50}$$

If we look at a representative plot of this expansion we see several items of interest, figure 1.5.

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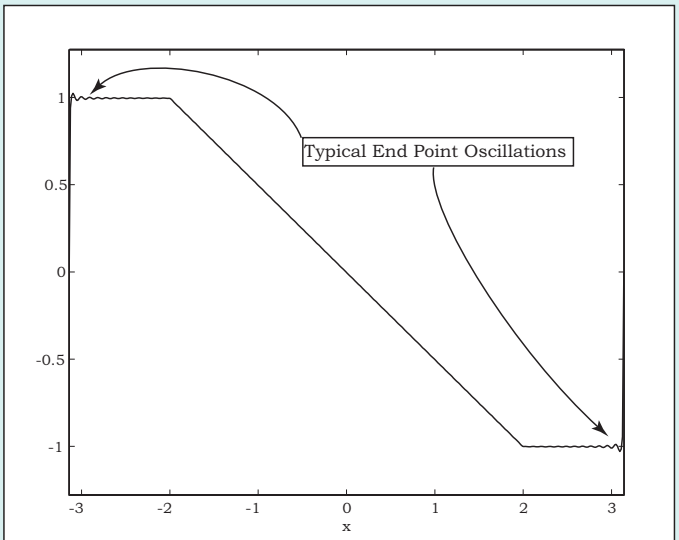


Figure 1.5: 500 Term Fourier series, equation 1.50

Even with 500 terms included in the series there seems to be some oscillation at both ends of the domain. This is due to the nature of our basis functions. A Fourier series is used in situations where the solution must fit a periodic function. As a result, outside our domain the function repeats itself, figure 1.6.

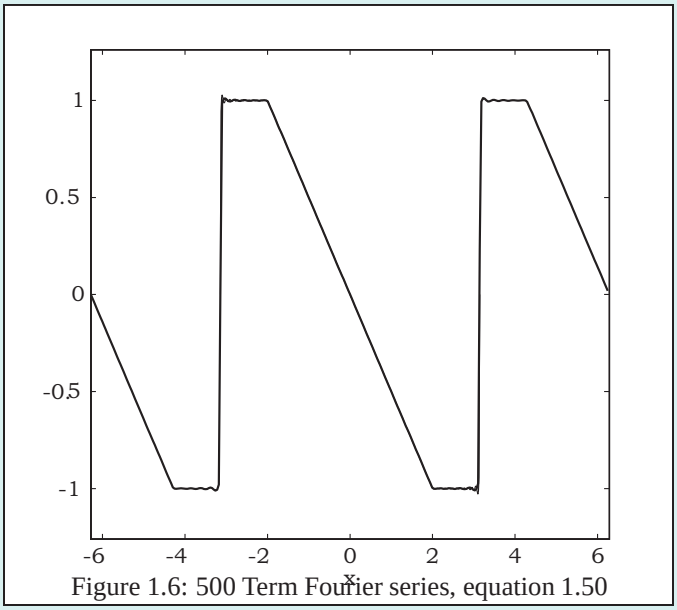


Figure 1.6: 500 Term Fourier series, equation 1.50

As we shall see in the next few sections, the choice for our basis functions will play a

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role in how accurate a truncated series represents our function is in the domain of interest. In addition, our basis functions will have an effect on what happens to our function outside the domain of interest. One may argue there is no value in what our basis functions do outside the domain of interest. However, as we can see in the Fourier series example the oscillations occurred at our endpoints due to the fact that the Fourier series produced a periodic function. This example along with plots are included in the Matlab script `FourierExample.m`.

1.2 Orthonormal Functions

Obtaining orthonormal functions from an orthogonal set is a very simple process. As we can see from the orthogonality relations shown in equation 1.3, every function in the series is orthogonal to every other function in the series. In other words, the function $\sin(nx)$ is orthogonal to all functions in the series except itself. The same is true for the $\cos(nx)$ term. In fact, the only way a function can be orthogonal to itself is if that function is zero everywhere. By looking directly at these self-integration statements in the Fourier series we notice the following,

$$\begin{aligned} \int_{-\pi}^{\pi} \sin(nx)\sin(nx)\,dx &= \pi \\ \int_{-\pi}^{\pi} \cos(nx)\cos(nx)\,dx &= \pi \\ \int_{-\pi}^{\pi} \left[\frac{1}{2}\right]\left[\frac{1}{2}\right] dx &= \frac{\pi}{2} \end{aligned} \tag{1.51}$$

If we altered our spanning set of functions, equation 1.43 by multiplying each term by a constant and forcing the resulting integration to unity, we would obtain an orthonormal basis function set.

$$\begin{aligned} \int_{-\pi}^{\pi} [A\sin(nx)]\,[A\sin(nx)]\,dx &= A^2\pi = 1 \\ \int_{-\pi}^{\pi} [B\cos(nx)]\,[B\cos(nx)]\,dx &= B^2\pi = 1 \\ \int_{-\pi}^{\pi} \left[\frac{C}{2}\right]\left[\frac{C}{2}\right] dx &= C^2\frac{\pi}{2} = 1 \end{aligned} \tag{1.52}$$

In order for these equations to be satisfied the function sets are altered:

$$\psi_n(x) = \left\{ \begin{array}{c} \psi_0(x) \\ \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \\ \psi_5(x) \\ \psi_6(x) \\ \vdots \end{array} \right\} = \left\{ \begin{array}{c} \frac{1}{\sqrt{2\pi}} \\ \frac{1}{\sqrt{\pi}}\sin(x) \\ \frac{1}{\sqrt{\pi}}\cos(x) \\ \frac{1}{\sqrt{\pi}}\sin(2x) \\ \frac{1}{\sqrt{\pi}}\cos(2x) \\ \frac{1}{\sqrt{\pi}}\sin(3x) \\ \frac{1}{\sqrt{\pi}}\cos(3x) \\ \vdots \end{array} \right\} \tag{1.53}$$

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With these orthonormal functions, our bilinear orthogonality relationship becomes.

$$\mathcal{B}(\psi_i, \psi_j) = \mathcal{B}(\psi_j, \psi_i) = \int_{-\pi}^{\pi} \psi_i(x)\,\psi_j(x)\,dx = \delta_{ij} \tag{1.54}$$

In review, any orthogonal set of functions may be recast into an orthonormal set of functions by simply scaling each term. It is not very often that such a process is needed. However, there are cases where such a process will help with respect to solution time especially in the finite element methods.

1.3 Fourier Series vs Monomial Series

To this point we have focused on Fourier series exclusively. The inevitable question is: Is there any other functions which would be appropriate to use? To answer this question, we look at a series representation of the $\sin(x)$ and $\cos(x)$ functions. A Taylor series expansion of each function reveals something interesting.

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+1}}{(2n+1)!} \tag{1.55}$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n}}{(2n)!} \tag{1.56}$$

These equations reveal a primal set of functions, often called monomials. These monomials may also be used as a spanning set in each of our functional spaces. For example, the monomials $1, x^2, x^4, x^6, \cdots$ will span space II. While the monomials $x, x^3, x^5, x^7, \cdots$ will span space III. Together the entire monomial set will span the functional space IV. Therefore, any function in space IV may be written as,

$$f(x) = \sum_{i=0}^{\infty} c_i x^i \tag{1.57}$$

There are several properties of these monomials worth noting at this point, first being orthogonality. It is obvious from the equation below that these monomials are not orthogonal.

$$\mathcal{B}(\psi_i, \psi_j) = \int_{x_L}^{x_R} x^i x^j dx = \int_{x_L}^{x_R} x^{(i+j)} dx = \left[\frac{x^{(i+j+1)}}{i+j+1}\right]_{x_L}^{x_R} \tag{1.58}$$

However, these monomials do span the functional space IV. In fact, *a spanning set of functions do not need to be orthogonal*. It is however, very convenient of we can generate a spanning set of functions that are orthogonal. It is possible to find several sets of polynomial basis functions that are orthogonal. Two such function sets are shown in the next few sections. In addition, we may even generate an orthogonal set of functions from nonorthogonal functions through a Gram-Schmidt orthonormalization process. This process is shown later in this chapter and will be used extensively in the weighted residual and finite element method chapters.

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1.4 Legendre Polynomials

One set of polynomials that are orthogonal by definition, originate from the solution of a Sturm-Liouville problem shown below.

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{\mathcal{P}_k\left(x\right)}{dx}\right]+k\left(k+1\right)\mathcal{P}_k\left(x\right)=0 \tag{1.59}$$

The Legendre polynomials, $\mathcal{P}_k\left(x\right)$, are eigenfunctions for this differential equation. The Legendre polynomial obey an interesting recurrence relationship shown in equations 1.60 to 1.62

$$\mathcal{P}_0\left(x\right)=1 \tag{1.60}$$

$$\mathcal{P}_1\left(x\right)=x \tag{1.61}$$

$$\mathcal{P}_{k+1}\left(x\right)=x\left[\frac{2k+1}{k+1}\right]\mathcal{P}_k\left(x\right)-\left[\frac{k}{k+1}\right]\mathcal{P}_{k-1}\left(x\right)\qquad\forall\quad k\geq 1 \tag{1.62}$$

Using this relationship we can easily write a subroutine to construct a polynomial set of any number of these functions by a simple loop. Another important property is the recursive derivative relationship shown in equation 1.63.

$$\frac{d\mathcal{P}_k\left(x\right)}{dx}=\frac{k\mathcal{P}_{k-1}\left(x\right)-kx\mathcal{P}_k\left(x\right)}{1-x^2} \tag{1.63}$$

Since the derivative of any polynomial may be written as a function of the Legendre functions themselves, a simple subroutine may be written to produce these derivatives. Of course the main reason for including these functions in this text is their orthogonality.

$$\mathcal{B}\left(\mathcal{P}_i,\mathcal{P}_j\right)=\int_{-1}^{1}\mathcal{P}_i\left(x\right)\mathcal{P}_j\left(x\right)dx=\frac{\delta_{ij}}{i+\frac{1}{2}} \tag{1.64}$$

Just as in the Fourier series representation of a function, these orthogonality relationship will be used to construct functional representations based on these Legendre polynomials. Equations 1.59 to 1.64 are proven in the Matlab file `SturmLiouvilleLegendre.m`. In this Matlab file, the recursion relationship, the differential relationship and eigenfunction test are made. An abbreviated list of the Legendre functions are shown below in equation 1.65.

$$\begin{aligned}
\mathcal{P}_0\left(x\right)&=1\\
\mathcal{P}_1\left(x\right)&=x\\
\mathcal{P}_2\left(x\right)&=\frac{1}{2}\left(3x^2-1\right)\\
\mathcal{P}_3\left(x\right)&=\frac{1}{2}\left(5x^3-3x\right)\\
\mathcal{P}_4\left(x\right)&=\frac{1}{8}\left(35x^4-30x^2+3\right)\\
\mathcal{P}_5\left(x\right)&=\frac{1}{8}\left(63x^5-70x^3+15x\right)
\end{aligned} \tag{1.65}$$

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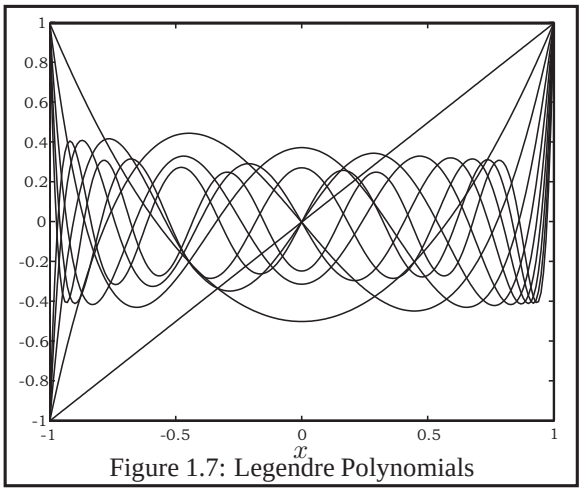


Figure 1.7: Legendre Polynomials

A plot of these functions are shown in figure 1.7. We will use these as well as other functions in the next several chapters. As we shall see in the finite element method, they are very useful enriching functions.

Example

In this section we shall use our Legendre polynomials to approximate the ramp function shown in figure 1.4. We will make use of our orthogonality relationship, equation 1.64, to determine the unknown constants, c_i. We begin by writing our ramp function as a Legendre series:

$$f(x) = \sum_{i=0}^{\infty} c_i \widetilde{\mathcal{P}}_i(x) \tag{1.66}$$

where the c_i are unknown constants. A very important fact to note at this point is the domain. From the plot shown in figure 1.4 the domain is $-\pi$ to π while the domain of orthogonality for Legendre polynomials is from -1 to $+1$. This is why we have list our Legendre polynomials as $\widetilde{\mathcal{P}}_j(x)$ instead of $\mathcal{P}_j(x)$. We now seek a domain transformation, mapping the domain $-\pi \leq x \leq \pi$ into $-1 \leq y \leq +1$. Such a mapping is very simple and may be expressed as,

$$x = \pi y \tag{1.67}$$

When $y = -1$, $x = -\pi$ and when $y = 1$, $x = \pi$. We transform our ramp function to the orthogonal domain, $-1 \leq y \leq +1$, and transform our results back to original domain, $-\pi \leq x \leq \pi$. This transformation will give a new function, shown in figure 1.8. Our new equation may be written as,

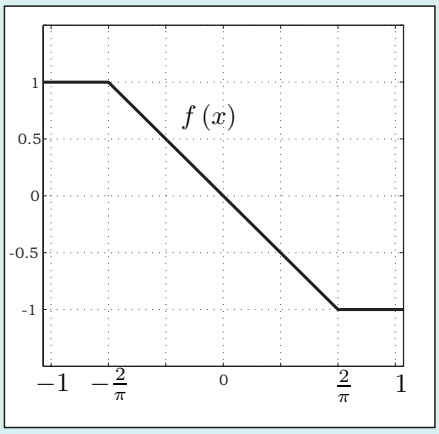


Figure 1.8: Transformed Ramp Function

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If we multiply our equation by any of our

$$\widetilde{f}(y) = \sum_{i=0}^{\infty} c_i \mathcal{P}_i(y) \tag{1.68}$$

Legendre polynomials, say $\mathcal{P}_j(x)$, and integrate over the domain we have,

$$\int_{-1}^{1} \widetilde{f}(y)\,\mathcal{P}_j(y)\,dx = \sum_{i=0}^{\infty} c_i \int_{-1}^{1} \mathcal{P}_i(y)\,\mathcal{P}_j(y)\,dy = \sum_{i=0}^{\infty} c_i \frac{\delta_{ij}}{i+\frac{1}{2}} \tag{1.69}$$

Or

$$c_j = \left(j+\frac{1}{2}\right)\int_{-1}^{1} \widetilde{f}(y)\,\mathcal{P}_j(y)\,dx \tag{1.70}$$

The integral must be broken into three sections,

$$c_j = \left(j+\frac{1}{2}\right)\left[\int_{-1}^{-2/\pi} \mathcal{P}_j(y)\,dx + \int_{-2/\pi}^{2/\pi} \left(-\frac{y}{\pi}\right)\mathcal{P}_j(y)\,dx - \int_{2/\pi}^{1} \mathcal{P}_j(y)\,dx\right] \tag{1.71}$$

Carrying out this integral and transforming our results we get the plot shown in figure 1.9.

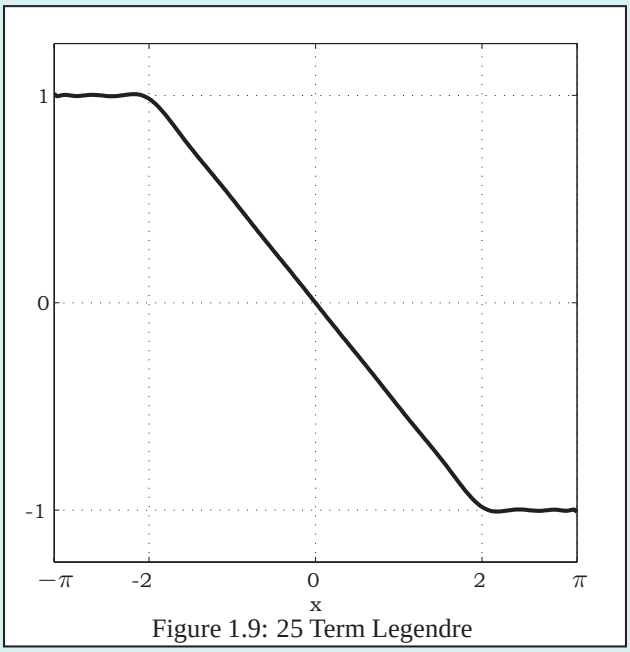


Figure 1.9: 25 Term Legendre

With just 25 terms this truncated series is very close to our original function. In addition this truncated series does not exhibit the end point oscillations associated with the Fourier series representation. If we extend our plot domain out and look at how the approximation behaves outside of our area of interest we see and interesting result, figure 1.10.

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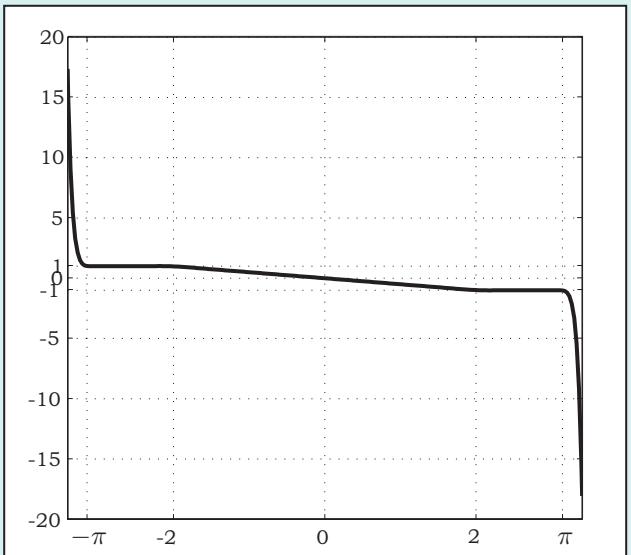


Figure 1.10: 25 Term Legendre

Outside the domain of interest, the function increases rapidly the further we move away from $|x| > \pi$. Remember the Fourier series was periodic outside our domain of interest. Since the Legendre polynomials themselves are not periodic, the representation of this function is not periodic and the end points of our domain exhibit no oscillations. In conclusion, the Matlab script `LegendreExample.m` contains the complete code for executing and plotting this example.

1.5 Tchebychev Polynomials

In this section we investigate the use of Tchebychev polynomials as basis functions. Tchebychev polynomials of the first kind come from the Sturm-Liouville differential equation of the form shown below.

$$\frac{d}{dx}\left[\sqrt{(1-x^2)}\frac{\mathcal{T}_k(x)}{dx}\right] + \frac{k}{\sqrt{(1-x^2)}}\mathcal{T}_k(x) = 0 \tag{1.72}$$

Just like Legendre polynomials, Tchebychev polynomials have nice recurrence relationships. These recurrence relationships can be used in a subroutine to produce a polynomial of any order very quickly. The recursive relationship for Tchebychev polynomials are shown in equations 1.73 to 1.75.

$$\mathcal{T}_0(x) = 1 \tag{1.73}$$

$$\mathcal{T}_1(x) = x \tag{1.74}$$

$$\mathcal{T}_{k+1}(x) = 2x\mathcal{T}_k(x) - \mathcal{T}_{k-1}(x) \qquad \forall\, k \geq 1 \tag{1.75}$$

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There are several relationships possible for the derivatives. However, we are interested in one which may be used in a subroutine. To this end we use equation 1.76

$$\frac{d\mathcal{T}_k(x)}{dx} = 2\mathcal{T}_{k-1}(x) + 2x\frac{d\mathcal{T}_{k-1}(x)}{dx} - \frac{d\mathcal{T}_{k-2}(x)}{dx} \qquad \forall\, k \geq 2 \tag{1.76}$$

The orthogonality relationship is a bit different than what we have see previously. The relationship is a weighted orthogonality, as shown in equation 1.77

$$\int_{-1}^{1} \mathcal{T}_i(x)\,\mathcal{T}_j(x)\,w(x)\,dx = \begin{cases} \pi\delta_{ij} & i = 0 \\ \frac{\pi}{2}\delta_{ij} & i > 0 \end{cases} \tag{1.77}$$

Where the weighting term is,

$$w(x) = \frac{1}{\sqrt{(1-x^2)}} \tag{1.78}$$

To force our function to be orthogonal in a traditional sense we could redefine our functions as,

$$\varphi_i(x) = \frac{\mathcal{T}_i(x)}{\sqrt[4]{(1-x^2)}} \tag{1.79}$$

Then our equations would become,

$$\mathcal{B}(\varphi_i, \varphi_j) = \int_{-1}^{1} \varphi_i(x)\,\varphi_j(x)\,dx = \begin{cases} \pi\delta_{ij} & i = 0 \\ \frac{\pi}{2}\delta_{ij} & i > 0 \end{cases} \tag{1.80}$$

However, we will not find much use for forcing the Tchebychev polynomials to be orthogonal in a traditional sense. This is primarily due to the use of a Gram-Schmidt orthonormalization process describe later in this chapter.

An expanded set of the Tchebychev polynomials functions are listed in equation 1.81 below.

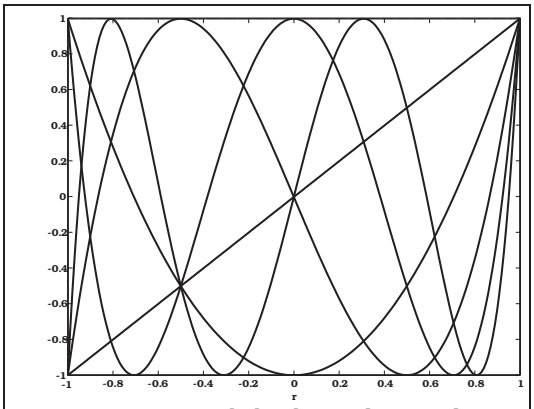


Figure 1.11: Tchebychev Polynomials

$$\begin{aligned} \mathcal{T}_0(x) &= 1 \\ \mathcal{T}_1(x) &= x \\ \mathcal{T}_2(x) &= -1 + 2x^2 \\ \mathcal{T}_3(x) &= -3x + 4x^3 \\ \mathcal{T}_4(x) &= 1 - 8x^2 + 8x^4 \\ \mathcal{T}_5(x) &= 5x - 20x^3 + 16x^5 \end{aligned} \tag{1.81}$$

A plot of Tchebychev polynomials is shown in figure 1.11. One particular thing to notice about Tchebychev polynomials is their magnitude. The maximum of all functions is unity. This is due to the fact that Tchebychev polynomials are generated from a trigonometric transformation. Interested reader may consult reference [**?**] as well as the reference therein.

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Example

In this section we will develop a Tchebychev polynomial series representation to our ramp function, shown in figure 1.8. We will make use of the weighted orthogonality relationship equation 1.77 to determine the constants in the series. If we write our function as a Tchebychev series we have,

$$\widetilde{f}(x) = \sum_{i=0}^{\infty} c_i \mathcal{T}_i(x) \tag{1.82}$$

where the c_i are unknown constants. We must now multiply this equation by both a weighting function, equation 1.78, as well as a representative Tchebychev polynomial. We then integrate over our transformed domain,

$$\int_{-1}^{1} w(x) \mathcal{T}_j(x) \widetilde{f}(x)\, dx = \sum_{i=0}^{\infty} c_i \int_{-1}^{1} \mathcal{T}_i(x) \mathcal{T}_j(x) w(x)\, dx \tag{1.83}$$

Using the orthogonality relationship we have,

$$c_j = A_j \int_{-1}^{1} w(x) \mathcal{T}_j(x) \widetilde{f}(x)\, dx \tag{1.84}$$

where A_j is defined as,

$$A_j = \begin{cases} \pi & \text{if } j = 0 \\ \frac{\pi}{2} & \text{if } j > 0 \end{cases} \tag{1.85}$$

Using Matlab to carry out the integration and polynomial construction we obtain a plot for a 25 term expansion shown in figure 1.12.

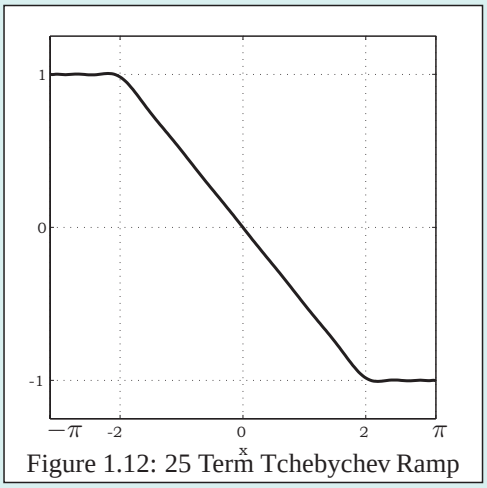


Figure 1.12: 25 Term Tchebychev Ramp

As in the Legendre polynomial expansion the Tchebychev series produces a very good solution with only 25 terms. In addition, outside of our domain behaves much like the Legendre series, figure 1.13.

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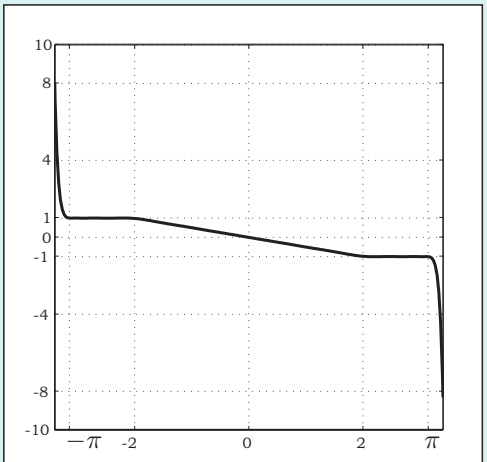


Figure 1.13: 25 Term Tchebychev Ramp

In conclusion, both Legendre and Tchebychev polynomials may be used to obtain polynomial representations of any given function. Orthogonality is a major tool used to determine the constants of this polynomial series. The process for using this orthogonality is exactly that used in Fourier series expansions of a function. Namely, multiplying the series by one of the orthogonal functions and integrating over the domain. The orthogonality relationship removes every term except the term of interest, from which our constants may be determined.

1.6 Polynomial based transformations

In this section, we include a simple yet very useful transformation between Legendre and monomial functions, as well as, between Tchebychev and monomial functions. These will be used in later sections to transform our system matrices between polynomial groups. As it turns out, these transformations may be used to move a matrix from one basis functions to any other basis functions. To begin with we observe the Legendre polynomials in equation 1.65. Expanding these terms out we have the transformation relationship,

$$\begin{Bmatrix} \mathcal{P}_0(x) \\ \mathcal{P}_1(x) \\ \mathcal{P}_2(x) \\ \mathcal{P}_3(x) \\ \mathcal{P}_4(x) \\ \mathcal{P}_5(x) \end{Bmatrix} = \begin{bmatrix} 1. & 0 & 0 & 0 & 0 & 0 \\ 0 & 1. & 0 & 0 & 0 & 0 \\ -0.5 & 0 & 1.5 & 0 & 0 & 0 \\ 0 & -1.5 & 0 & 2.5 & 0 & 0 \\ 0.375 & 0 & -3.75 & 0 & 4.375 & 0 \\ 0 & 1.875 & 0 & -8.75 & 0 & 7.875 \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \\ x^5 \end{Bmatrix} \tag{1.86}$$

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Likewise, the Tchebychev polynomials, equation 1.81, have a similar transforation matrix.

$$\begin{Bmatrix} \mathcal{T}_0(x) \\ \mathcal{T}_1(x) \\ \mathcal{T}_2(x) \\ \mathcal{T}_3(x) \\ \mathcal{T}_4(x) \\ \mathcal{T}_5(x) \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & 0 & 0 \\ 0 & -3 & 0 & 4 & 0 & 0 \\ 1 & 0 & -8 & 0 & 8 & 0 \\ 0 & 5 & 0 & -20 & 0 & 16 \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \\ x^5 \end{Bmatrix} \tag{1.87}$$

If we designate our monomial equation basis functions as,

$$\mathcal{M}_i(x) = x^i \tag{1.88}$$

We may rewrite equation 1.87 as,

$$\begin{Bmatrix} \mathcal{T}_0(x) \\ \mathcal{T}_1(x) \\ \mathcal{T}_2(x) \\ \mathcal{T}_3(x) \\ \mathcal{T}_4(x) \\ \mathcal{T}_5(x) \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & 0 & 0 \\ 0 & -3 & 0 & 4 & 0 & 0 \\ 1 & 0 & -8 & 0 & 8 & 0 \\ 0 & 5 & 0 & -20 & 0 & 16 \end{bmatrix} \begin{Bmatrix} \mathcal{M}_0(x) \\ \mathcal{M}_1(x) \\ \mathcal{M}_2(x) \\ \mathcal{M}_3(x) \\ \mathcal{M}_4(x) \\ \mathcal{M}_5(x) \end{Bmatrix} \tag{1.89}$$

To generalized these relationships it is convenient to write these equations in symbolic form:

$$\{\mathcal{P}\} = \left[T^{\mathcal{PM}}\right]\{\mathcal{M}\} \tag{1.90}$$

and,

$$\{\mathcal{T}\} = \left[T^{\mathcal{TM}}\right]\{\mathcal{M}\} \tag{1.91}$$

Since each of these transformations are invertible, we may determine a transformation between Tchebychev and Legendre polynomials by inverting one of the transformations and replacing the monomials in the other equation by this inversion.

$$\{\mathcal{M}\} = \left[T^{\mathcal{TM}}\right]^{-1}\{\mathcal{T}\} \tag{1.92}$$

Substituting equation 1.92 into equation 1.90 we obtain,

$$\{\mathcal{P}\} = \left[T^{\mathcal{PM}}\right]\left[T^{\mathcal{TM}}\right]^{-1}\{\mathcal{T}\} \tag{1.93}$$

or

$$\{\mathcal{P}\} = \left[T^{\mathcal{PT}}\right]\{\mathcal{T}\} \tag{1.94}$$

where,

$$\left[T^{\mathcal{PT}}\right] = \left[T^{\mathcal{PM}}\right]\left[T^{\mathcal{TM}}\right]^{-1} \tag{1.95}$$

In addition, as reverse transformation is possible:

$$\left[T^{\mathcal{TP}}\right] = \left[T^{\mathcal{PT}}\right]^{-1} = \left[T^{\mathcal{TM}}\right]\left[T^{\mathcal{PM}}\right]^{-1} \tag{1.96}$$

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One interesting property of both transformation matrices, equation 1.86 and 1.87, may been seen by observing the numbers above the diagonal. Every term is zero above this diagonal. this results from one fact, the highest order of Legendre and Tchebychev polynomials is the same as the order of the monomial. Mathematically, this may be stated as, the highest order in the polynomials, $\mathcal{P}_i(x)$ and $\mathcal{T}_i(x)$ is the same as in the monomial, $\mathcal{M}_i(x) = x^i$. This property will be very important to us later and will be used to construct hierarchial basis function in the finite element method.

1.6.1 Series Transformation

We have already seen a series may be written as,

$$f(x) = \sum_{i=0}^{\infty} c_i \mathcal{T}_i(x) \tag{1.97}$$

In addition, our transformations may be written in summation form as,

$$\mathcal{T}_i(x) = \sum_{j=0}^{\infty} T_{ij}^{\mathcal{T}\mathcal{M}} \mathcal{M}_j \tag{1.98}$$

This equation is exactly the same, in indexed notation, as equation 1.87 and 1.91. Since there is one free index, i, in this equation, the equation is a vector equation. In other words, the left hand side of equation 1.98, 1.87 and 1.91 is a vector, and therefore the right hand side must be a vector as well. If we multiply this vector equation by c_i,

$$c_i \mathcal{T}_i(x) = c_i \sum_{j=0}^{\infty} T_{ij}^{\mathcal{T}\mathcal{M}} \mathcal{M}_j \tag{1.99}$$

and sum over all possible terms,

$$\sum_{i=0}^{\infty} c_i \mathcal{T}_i(x) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_i T_{ij}^{\mathcal{T}\mathcal{M}} \mathcal{M}_j \tag{1.100}$$

Since there is no free indices on either side of the equation, this equation is now a scalar and is exactly the same as equation 1.97. Hence,

$$f(x) = \sum_{i=0}^{\infty} c_i \mathcal{T}_i(x) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_i T_{ij}^{\mathcal{T}\mathcal{M}} \mathcal{M}_j = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} c_i T_{ij}^{\mathcal{T}\mathcal{M}} \mathcal{M}_j \tag{1.101}$$

Note: reversing the summation gives the same result. Our original equation may be written as a series of monomials as well,

$$f(x) = \sum_{j=0}^{\infty} a_j \mathcal{M}_j \tag{1.102}$$

Noticing the similarities between these two equations, 1.101 and 1.102, highlighted below. The terms below in boxes are equal.

$$f(x) = \sum_{i=0}^{\infty} c_i \mathcal{T}_i(x) = \sum_{j=0}^{\infty} \boxed{\sum_{i=0}^{\infty} c_i T_{ij}^{\mathcal{T}\mathcal{M}}} \mathcal{M}_j = \sum_{j=0}^{\infty} \boxed{a_j} \mathcal{M}_j \tag{1.103}$$

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Or

$$\sum_{i=0}^{\infty} c_i T_{ij}^{\mathcal{TM}} = a_j \tag{1.104}$$

Or, in matrix form.

$$\{a\} = \left[T^{\mathcal{TM}}\right]^T \{c\} \tag{1.105}$$

In review: If we obtained the constants for a Tchebychev series, c_i's, we may use the transformation matrix between the Tchebychev and monomial series terms, $T_{ij}^{\mathcal{TM}}$, to determine the constants based on the monomial series a_i's. This statement also applies in reverse, given the constants a_i's we may use this transformation to obtain the c_i's.

Example

Given a function,

$$f(x) = 5 + 2x^2 + 8x^3 + 10x^4 \tag{1.106}$$

Monomial to Tchebychev Transformation

We wish to express this in terms of a Tchebychev polynomial. At present it is expressed as a monomial series with the constants.

$$\begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix} = \begin{Bmatrix} 5 \\ 0 \\ 2 \\ 8 \\ 10 \end{Bmatrix} \tag{1.107}$$

Our equation 1.105 may be written as

$$\{c\} = \left[T^{\mathcal{TM}}\right]^{-T} \{a\} \tag{1.108}$$

Or written in matrix form, using the transpose of the inverse of the transformation matrix shown in equation 1.87,

$$\begin{Bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0.5 & 0 & 0.375 & 0 \\ 0 & 1 & 0 & 0.75 & 0 & 0.625 \\ 0 & 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.25 & 0 & 0.3125 \\ 0 & 0 & 0 & 0 & 0.125 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.0625 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{Bmatrix} \tag{1.109}$$

Or for our specific case,

$$\begin{Bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0.5 & 0 & 0.375 & 0 \\ 0 & 1 & 0 & 0.75 & 0 & 0.625 \\ 0 & 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.25 & 0 & 0.3125 \\ 0 & 0 & 0 & 0 & 0.125 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.0625 \end{bmatrix} \begin{Bmatrix} 5 \\ 0 \\ 2 \\ 8 \\ 10 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 9.75 \\ 6 \\ 6 \\ 2 \\ 1.25 \\ 0 \end{Bmatrix} \tag{1.110}$$

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Using this result and equations 1.81 our function may be written as,

$$\boxed{f(x) = 9.75 \times \mathcal{T}_0(x) + 6 \times \mathcal{T}_1(x) + 6 \times \mathcal{T}_2(x) + 2 \times \mathcal{T}_3(x) + 1.25 \times \mathcal{T}_4(x)} \quad (1.111)$$

Expanding the Tchebychev polynomials out we have,

$$\begin{aligned} f(x) =& 9.75(1) + 6(x) + 6\left(-1 + 2x^2\right) + \\ & 2\left(-3x + 4x^3\right) + 1.25\left(1 - 8x^2 + 8x^4\right) \end{aligned} \quad (1.112)$$

Carefully extracting the coefficients of each term gives,

$$\begin{aligned} 1: &\quad 9.75 - 6 + 1.25 = 5 \\ x: &\quad 6 - 3 \times 2 = 0 \\ x^2: &\quad 2 \times 6 - 1.25 \times 8 = 2 \\ x^3: &\quad 2 \times 4 = 8 \\ x^4: &\quad 1.25 \times 8 = 10 \end{aligned} \quad (1.113)$$

Comparing these constants with original coefficients in equation 1.106 we may verify our transformation is indeed correct.

Monomial to Legendre Transformation

We now wish to perform the same operation with Legendre functions. Using the transpose of the inverse of the transformation matrix shown in equation 1.86 we may determined the Legendre polynomial coefficients of equation 1.106 as,

$$\begin{Bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0.3333 & 0 & 0.2 & 0 \\ 0 & 1 & 0 & 0.6 & 0 & 0.4286 \\ 0 & 0 & 0.6667 & 0 & 0.5714 & 0 \\ 0 & 0 & 0 & 0.4 & 0 & 0.4444 \\ 0 & 0 & 0 & 0 & 0.2286 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.127 \end{bmatrix} \begin{Bmatrix} 5 \\ 0 \\ 2 \\ 8 \\ 10 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 7.6667 \\ 4.8 \\ 7.0476 \\ 3.2 \\ 2.2857 \\ 0 \end{Bmatrix} \quad (1.114)$$

Using this result and equation 1.65 our function may be written as,

$$\boxed{\begin{aligned} f(x) =& 7.6667 \times \mathcal{P}_0(x) + 4.8 \times \mathcal{P}_1(x) + 7.0476 \times \mathcal{P}_2(x) \\ & + 3.2 \times \mathcal{P}_3(x) + 2.2857 \times \mathcal{P}_4(x) \end{aligned}} \quad (1.115)$$

Expanding the Legendre polynomials out we have,

$$\begin{aligned} f(x) =& 7.6667(1) + 4.8(x) + \frac{7.0476}{2}\left(3x^2 - 1\right) + \\ & \frac{3.2}{2}\left(5x^3 - 3x\right) + \frac{2.2857}{8}\left(35x^4 - 30x^2 + 3\right) \end{aligned} \quad (1.116)$$

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Carefully extracting the coefficients of each term gives,

$$\begin{aligned} 1: &\quad 7.6667 - \frac{7.0476}{2} + 3 \times \frac{2.2857}{8} = 5 \\ x: &\quad 4.8 - 3 \times \frac{3.2}{2} = 0 \\ x^2: &\quad 3 \times \frac{7.0476}{2} - 30 \times \frac{2.2857}{8} = 2 \\ x^3: &\quad 5 \times \frac{3.2}{2} = 8 \\ x^4: &\quad 35 \times \frac{2.2857}{8} = 10 \end{aligned} \tag{1.117}$$

As we can see the transformation matrix gives different constants, equations 1.111 compared to equation 1.115. However, the polynomial is represented exactly in both cases. We shall use these types of transformations extensively later in this text.

1.7 Gram-Schmidt Orthonormalization

There is significant advantages to having basis functions which are orthonormal. As we have seen previously, using orthogonal functions which span a particular functional space allows us to express any function in that space easily. Our eventual usage for these basis functions will be to obtain solutions to partial differential equations in multidimensions. As it turns out, the use of orthonormal basis functions can significantly reduce the amount of computational effort required to obtain an accurate numerical solution to these partial differential equations. Central to the method that follows is the existence of a bilinear form. To this point the bilinear form has been the traditional orthogonality definition,

$$\mathcal{B}\left(\psi_i, \psi_j\right) = \int_{x_l}^{x_r} \psi_i\left(x\right) \psi_j\left(x\right) dx \tag{1.118}$$

However, other definition of this bilinear form are acceptable, see problem 20,

$$\mathcal{B}\left(\psi_i, \psi_j\right) = \int_{x_l}^{x_r} \frac{d\psi_i}{dx} \frac{d\psi_j}{dx} dx \tag{1.119}$$

We now describe a process of generating an orthonormal spanning functional set from any non-orthogonal spanning set. The process is called Gram-Schmidt orthonormalization method. This method date back to the late 1800's and can be found in most linear algebra and functional analysis textbooks. The Gram-Schmidt process can be stated as follows.

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Given a set of linearly independent functions $\{\phi_1, \phi_2, \phi_3, \phi_4, \ldots, \phi_n\}$ which span some functional space $\mathcal{L}$. We may construct a an orthonormal set of functions $\{\psi_1, \psi_2, \psi_3, \psi_4, \ldots, \psi_n\}$ using the Gram-Schmidt process which in turn will also span this functional space $\mathcal{L}$. The process is defined below:

$$\psi_1 = \frac{\phi_1}{\sqrt{\mathcal{B}(\phi_1, \phi_1)}}$$

$$\widetilde{\psi}_2 = \phi_2 - \psi_1 \mathcal{B}(\psi_1, \phi_2)$$

$$\psi_2 = \frac{\widetilde{\psi}_2}{\sqrt{\mathcal{B}\left(\widetilde{\psi}_2, \widetilde{\psi}_2\right)}}$$

$$\widetilde{\psi}_3 = \phi_3 - \psi_1 \mathcal{B}(\psi_1, \phi_3) - \psi_2 \mathcal{B}(\psi_2, \phi_3)$$

$$\psi_3 = \frac{\widetilde{\psi}_3}{\sqrt{\mathcal{B}\left(\widetilde{\psi}_3, \widetilde{\psi}_3\right)}}$$

$$\widetilde{\psi}_4 = \phi_4 - \psi_1 \mathcal{B}(\psi_1, \phi_4) - \psi_2 \mathcal{B}(\psi_2, \phi_4) - \psi_3 \mathcal{B}(\psi_3, \phi_4)$$

$$\psi_4 = \frac{\widetilde{\psi}_4}{\sqrt{\mathcal{B}\left(\widetilde{\psi}_4, \widetilde{\psi}_4\right)}}$$

$$\vdots$$

$$\widetilde{\psi}_n = \phi_n - \left[\sum_{k=1}^{n-1} \psi_k \mathcal{B}(\psi_k, \phi_n)\right]$$

$$\psi_n = \frac{\widetilde{\psi}_n}{\sqrt{\mathcal{B}\left(\widetilde{\psi}_n, \widetilde{\psi}_n\right)}}$$

This process may seem overly complex, yet, it is very mechanical in nature. As a result, this process may be placed in a program and carried out by the computer. To solidify this concept, we will show an example using the monomials from the last section.

Example

Using the notation from the Gram-Schmidt process above, we may write the monomial basis functions as,

$$\phi_n(x) = \left\{ \begin{array}{c} \phi_1(x) \\ \phi_2(x) \\ \phi_3(x) \\ \phi_4(x) \\ \phi_5(x) \\ \phi_6(x) \end{array} \right\} = \left\{ \begin{array}{c} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \\ x^5 \end{array} \right\} \tag{1.120}$$

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We seek an orthonormal set of basis functions from this set of 6 functions. However, we must first define our bilinear functional $\mathcal{B}(\phi_i, \phi_j)$. Although we are not limited to this bilinear functional, we shall use the standard orthogonality relationship,

$$\mathcal{B}(\phi_i, \phi_j) = \int_{x_L}^{x_R} \phi_i(x)\, \phi_j(x)\, dx \tag{1.121}$$

Notice the limits of the integration are x_L and x_R. The limits in the Fourier series section is a special case. The limits of integration on the bilinear functional $\mathcal{B}(\phi_1, \phi_1)$ depend entirely on the range of the problem we are interested. We shall pick the range $0 \leq x \leq 1$ for simplicity. We may also note at this point, the bilinear functional is not defined in the Gram-Schmidt process. As a result, we must define our bilinear functional, equation 1.121. We could have defined it differently, see problems at the end of this chapter.

Step 1 - Gram-Schmidt

Using our bilinear functional, we perform step one of the process,

$$\mathcal{B}(\phi_1, \phi_1) = \int_0^1 dx = [x]_0^1 = 1 \tag{1.122}$$

From this result we can easily determine the first function as,

$$\psi_1 = \frac{\phi_1}{\sqrt{\mathcal{B}(\phi_1, \phi_1)}} = \frac{1}{\sqrt{\mathcal{B}(\phi_1, \phi_1)}} = \frac{1}{\sqrt{1}} = 1 \tag{1.123}$$

Step 2 - Gram-Schmidt

Next we must perform an orthogonalization step,

$$\begin{aligned}
\widetilde{\psi}_2 =& \phi_2 - \psi_1 \mathcal{B}(\psi_1, \phi_2) \\
=& \phi_2 - \psi_1 \int_0^1 \psi_1(x)\, \phi_2(x)\, dx \\
=& x - (1) \int_0^1 (1)\, x dx \\
=& x - (1) \left[\frac{x^2}{2}\right]_0^1 \\
=& x - \frac{1}{2}
\end{aligned} \tag{1.124}$$

This function is orthogonal to the first generated function, as we shall see in a moment. However, the function is not normalized to unity. This occurs in the next step.

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Step 3 - Gram-Schmidt

We now seek to normalize our second generated function. From the Gram-Schmidt orthonormalization process we have,

$$\begin{aligned}\psi_2 =& \frac{\widetilde{\psi}_2}{\sqrt{\mathcal{B}\left(\widetilde{\psi}_2, \widetilde{\psi}_2\right)}} \\ =& \frac{\left(x-\frac{1}{2}\right)}{\sqrt{\int_0^1 \left(x-\frac{1}{2}\right)\left(x-\frac{1}{2}\right) dx}} \\ =& \frac{\left(x-\frac{1}{2}\right)}{\sqrt{\frac{1}{12}}} \\ =& \sqrt{12}\left(x-\frac{1}{2}\right)\end{aligned} \tag{1.125}$$

The question inevitably arises; is this function orthogonal to the first function? To test this we look at this statement,

$$\begin{aligned}\mathcal{B}\left(\psi_1, \psi_2\right) =& \int_0^1 (1)\left(x-\frac{1}{2}\right) dx \\ =& \left[\frac{x^2}{2} - \frac{x}{2}\right]_0^1 \\ =& 0\end{aligned} \tag{1.126}$$

In addition, each function's norm is unity, thus the term orthonormal vs orthogonal.

$$\mathcal{B}\left(\psi_1, \psi_1\right) = \mathcal{B}\left(\psi_2, \psi_2\right) = 1 \tag{1.127}$$

Step 4 - Gram-Schmidt

We proceed to step four, determining the unscaled third function of the orthonormal set.

$$\begin{aligned}\widetilde{\psi}_3 =& \phi_3 - \psi_1 \mathcal{B}\left(\psi_1, \phi_3\right) - \psi_2 \mathcal{B}\left(\psi_2, \phi_3\right) \\ =& x^2 - (1)\int_0^1 (1)\, x^2 dx - \sqrt{12}\left(x-\frac{1}{2}\right)\int_0^1 \sqrt{12}\left(x-\frac{1}{2}\right) x^2 dx \\ =& x^2 - \left[\frac{x^3}{3}\right]_0^1 - 12\left(x-\frac{1}{2}\right)\left[\frac{x^4}{4} - \frac{x^3}{6}\right]_0^1 \\ =& x^2 - x + \frac{1}{6}\end{aligned} \tag{1.128}$$

This unscaled term should be orthogonal to both the first and second function. The reader may verify that indeed the following is true,

$$\mathcal{B}\left(\psi_1, \widetilde{\psi}_3\right) = \mathcal{B}\left(\psi_2, \widetilde{\psi}_3\right) = 0 \tag{1.129}$$

However, the function itself must now be scaled, as seen by the following equation.

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$$\mathcal{B}\left(\widetilde{\psi}_3, \widetilde{\psi}_3\right) = \frac{1}{180} \tag{1.130}$$

Step 5 - Gram-Schmidt

Normalizing this function may be done by use of step 5's equation.

$$\psi_3 = \frac{\widetilde{\psi}_3}{\sqrt{\mathcal{B}\left(\widetilde{\psi}_3, \widetilde{\psi}_3\right)}} \tag{1.131}$$

Therefore the function becomes,

$$\psi_3 = \sqrt{180}\left(x^2 - x + \frac{1}{6}\right) \tag{1.132}$$

Final Result - Gram-Schmidt

If we continue through all the functions in our set, equation 1.120, we generate a new set of orthonormal functions listed in equation 1.134. Although these functions are more complex than our monomials there are some interesting properties that may be observed. One property is the highest order in each term. For example, our third term was generate from the first, second and third term $(1, x, x^2)$ in our original set of monomial functions. As a result, this third term's highest order is second order. Another property is the possible existence of all order terms up to the order of the present term. For example, the third term contains is a linear combination of all three terms in the original monomial set, $(1, x, x^2)$. This may not always be the case, depending on what functions we start with, the ϕ_i's. Mathematically, both of these properties may be written as,

$$\begin{aligned} \psi_3(x) =& C_1\phi_1(x) + C_2\phi_2(x) + C_3\phi_3(x) \\ =& C_1 + C_2 x + C_3 x^2 \end{aligned} \tag{1.133}$$

All six basis functions may be written as,

$$\left\{\begin{array}{c} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \\ \psi_5(x) \\ \psi_6(x) \end{array}\right\} = \left\{\begin{array}{c} 1 \\ 2\,(x - 1/2)\sqrt{3} \\ 6\,\left(x^2 + 1/6 - x\right)\sqrt{5} \\ 20\,\left(x^3 - 1/20 + 3/5\,x - 3/2\,x^2\right)\sqrt{7} \\ 210\,x^4 + 3 - 60\,x + 270\,x^2 - 420\,x^3 \\ 252\,\left(x^5 - \frac{1}{252} + \frac{5}{42}\,x - 5/6\,x^2 + \frac{20}{9}\,x^3 - 5/2\,x^4\right)\sqrt{11} \end{array}\right\} \tag{1.134}$$

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Or in matrix form this may be written as,

$$\begin{Bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \end{Bmatrix} = \begin{bmatrix} 1.00 & 0 & 0 & 0 & 0 & 0 \\ -1.73 & 3.46 & 0 & 0 & 0 & 0 \\ 2.23 & -13.41 & 13.41 & 0 & 0 & 0 \\ -2.64 & 31.74 & -79.37 & 52.91 & 0 & 0 \\ 3.00 & -60.00 & 270 & -420 & 210 & 0 \\ -3.31 & 99.49 & -696.49 & 1857.31 & -2089.5 & 835.79 \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \\ x^5 \end{Bmatrix} \tag{1.135}$$

Matlab Script — Gram-Schmidt Orthonormalization Process

Since the Gram-Schmidt process is well defined and mechanical in nature, a Matlab script may be used to create an orthonormal set of basis functions from a nonorthogonal basis function set. Such a Matlab script is listed in figure 1.14. This script will take our original basis functions, equation 1.120, and generate an orthonormal basis function set. A complete description of the Matlab script follows.

The first five lines of the script are comments, as well as, a function descriptor in line 1. Line 6 creates a symbolic variable x for use in our integrations. Line 7 creates our original basis functions, equation 1.120. While line 8 scales the first function such that its bilinear form becomes unity, step 1 in the Gram-Schmidt process. Line 9 determines the number of terms in the functional set. Line 10 to 16 loop through the remaining Gram-Schmidt steps. Line 11 takes care of the portion of the Gram-Schmidt process shown in equation 1.136. Note, this equation was taken from the Gram-Schmidt definition.

$$\widetilde{\psi}_n = \underbrace{\phi_n}_{\text{Line 11}} - \underbrace{\left[\sum_{k=1}^{n-1} \psi_k \mathcal{B}\left(\psi_k, \phi_n\right)\right]}_{\text{Lines 12,13,14}} \tag{1.136}$$

Line 15 normalizes the function to unity with respect to the bilinear form, equation 1.137.

$$\psi_n = \frac{\widetilde{\psi}_n}{\sqrt{\mathcal{B}\left(\widetilde{\psi}_n, \widetilde{\psi}_n\right)}} \tag{1.137}$$

The transformation matrix between our new orthonormal basis functions and monomials, equation 1.135, is calculated on lines 18 to 20. Lines 27 to 31 and line 33 builds and outputs a matrix based on the bilinear form, equation 1.138.

$$M_{ij} = \mathcal{B}\left(\psi_i, \psi_j\right) = \int_0^1 \psi_i\left(x\right) \psi_j\left(x\right) dx \tag{1.138}$$

This matrix will be the identity matrix if we have accomplished our goal of producing an orthonormal set of functions. Lines 37 and 38 define a Matlab function for determining the bilinear integral given two functions as inputs.

The Gram-Schmidt process is the same regardless of the original basis functions used, line 7. Given another set of basis functions, line 7, or an alternate definition of our bilinear form, $\mathcal{B}\left(\psi_i, \psi_j\right)$, only two lines need be altered to make this script work on any case, line 7 and 38.

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Line 7 is the original set of functions we wish to produce our orthonormal set from. Line 38 is the integral defining our bilinear form. The limits on our example are from 0 to 1. Other limits may be desired based on our domain. An altered form of this script will be used later to generate orthonormal basis functions on the fly.

```
 1 function GramSchmidt
 2 %
 3 % Gram-Schmidt Ortho-normalization
 4 % Written By: Dr. Fithen
 5 %
 6 syms x
 7 phi=[1;x;x^2;x^3;x^4;x^5]
 8 psi(1)=phi(1)/sqrt(B(phi(1),phi(1)))
 9 N=length(phi)
10 for i=2:N
11     psi(i,1)=phi(i)
12     for j=1:i-1
13         psi(i,1)=psi(i,1)-psi(j,1)*B(psi(j,1),phi(i))
14     end
15     psi(i,1)=psi(i,1)/sqrt(B(psi(i,1),psi(i,1)))
16 end
17 % Lets determine the transformation matrix to monomials
18 for i=N:-1:1
19     vector=sym2poly(psi(i));
20     T(i,1:i)=flipdim(vector,2);
21 end
22 %
23 % Test orthogonality, Build a matrix
24 % This matrix will be the identity if our
25 % functions are orthonormal!
26 %
27 for i=1:N
28     for j=1:N
29         M(i,j)=B(psi(i,1),psi(j,1));
30     end
31 end
32 % Print/Display the matrix...
33 M
34 %
35 % Bilinear functional
36 %
37 function n=B(fi,fj)
38 n=int(fi*fj,0,1);
```

Figure 1.14: Gram-Schmidt Matlab Listing

1.8 First Fundamental Lemma of Variational Calculus

This section is a major reason including the first chapter. This theorem is the basis for both the weighted residual and finite element methods that follow throughout this text and other text as well. We have seen that a basis function set that spans a space may be either orthogonal or non-orthogonal. However, the Gram-Schmidt process can turn any non-orthogonal set of functions into an orthonormal set of functions. Since any set of basis functions may be converted into an orthonormal basis function set, we will begin this theorem with a set of orthonormal functions $\psi_i(x)$ which span space $\mathcal{L}$. We may represent any function, $\mathcal{R}(x)$, in

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a functional space $\mathcal{L}$ as a linear combination of these $\psi(x)$'s.

$$\mathcal{R}(x) = \sum_{i=1}^{\infty} a_i \psi_i(x) \tag{1.139}$$

Where the a_i's are constants. We now propose that our function $\mathcal{R}(x)$ is orthogonal to an infinite number of linearly independent functions each of which exist in space $\mathcal{L}$. For simplicity, we pick the infinite set of functions as the $\psi_i(x)$'s. We could have picked another set of functions. Since the function set $\psi_i(x)$ spans the functional space it is guaranteed to be an infinite set of linearly independent functions in space $\mathcal{L}$. In addition, any other set of functions which span the space may be transformed into this orthonormal set via Gram-Schmidt orthonormalization process. As a result, we may pick the function set, $\psi_i(x)$, without losing any generalization to our theorem.

Mathematically our orthogonality statement may be written as,

$$\int_{x_L}^{x_R} \mathcal{R}(x)\,\psi_i(x)\,dx = 0 \qquad \forall \quad i = 1, 2, 3, \cdots, \infty \tag{1.140}$$

From this equation, we see there is an infinite number of orthogonality equations, all of which must be satisfied.

$$\begin{aligned}
\int_{x_L}^{x_R} \mathcal{R}(x)\,\psi_1(x)\,dx &= 0 \\
\int_{x_L}^{x_R} \mathcal{R}(x)\,\psi_2(x)\,dx &= 0 \\
\int_{x_L}^{x_R} \mathcal{R}(x)\,\psi_3(x)\,dx &= 0 \\
&\vdots \\
\int_{x_L}^{x_R} \mathcal{R}(x)\,\psi_\infty(x)\,dx &= 0
\end{aligned} \tag{1.141}$$

We now insert equation 1.139 into 1.141 to obtain:

$$\begin{aligned}
\int_{x_L}^{x_R} \sum_{i=1}^{\infty} a_i \psi_i(x)\,\psi_1(x)\,dx &= 0 \\
\int_{x_L}^{x_R} \sum_{i=1}^{\infty} a_i \psi_i(x)\,\psi_2(x)\,dx &= 0 \\
\int_{x_L}^{x_R} \sum_{i=1}^{\infty} a_i \psi_i(x)\,\psi_3(x)\,dx &= 0 \\
&\vdots \\
\int_{x_L}^{x_R} \sum_{i=1}^{\infty} a_i \psi_i(x)\,\psi_\infty(x)\,dx &= 0
\end{aligned} \tag{1.142}$$

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Interchanging summations with integrations we obtain,

$$\begin{aligned}
\sum_{i=1}^{\infty} a_i \int_{x_L}^{x_R} \psi_i(x)\,\psi_1(x)\,dx &= 0 \\
\sum_{i=1}^{\infty} a_i \int_{x_L}^{x_R} \psi_i(x)\,\psi_2(x)\,dx &= 0 \\
\sum_{i=1}^{\infty} a_i \int_{x_L}^{x_R} \psi_i(x)\,\psi_3(x)\,dx &= 0 \\
&\vdots \\
\sum_{i=1}^{\infty} a_i \int_{x_L}^{x_R} \psi_i(x)\,\psi_\infty(x)\,dx &= 0
\end{aligned} \tag{1.143}$$

Remember by definition, an orthonormal set of functions obey the following relationship:

$$\int_{x_L}^{x_R} \psi_i(x)\,\psi_j(x)\,dx = \delta_{ij} \tag{1.144}$$

Using this definition in our infinite set of equations, 1.143, we obtain,

$$\begin{aligned}
\sum_{i=1}^{\infty} a_i\delta_{i1} &= a_1 = 0 \\
\sum_{i=1}^{\infty} a_i\delta_{i2} &= a_2 = 0 \\
\sum_{i=1}^{\infty} a_i\delta_{i3} &= a_3 = 0 \\
&\vdots \\
\sum_{i=1}^{\infty} a_i\delta_{i\infty} &= a_\infty = 0
\end{aligned} \tag{1.145}$$

Which results in all the constants a_i's being zero. As a result, the our function, $\mathcal{R}(x)$, is zero. Therefore first fundamental lemma of variational calculus may be stated as,

Given:

- A function, $\mathcal{R}(x)$, which exist in functional space $\mathcal{L}$
- A set of functions, $\psi_i(x)$, which span functional space $\mathcal{L}$.

If our function, $\mathcal{R}(x)$, is orthogonal to every function, $\psi_i(x)$, then the function $\mathcal{R}(x)$ is zero everywhere in the domain of orthogonality. Mathematically this may be written as,

$$\lim_{i\to\infty}\left[\int_{x_l}^{x_r} \mathcal{R}(x)\,\psi_i(x)\,dx = 0 \quad \forall\ i\right] \Longrightarrow \mathcal{R}(x) = 0$$

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Later we will use this theorem to obtain approximate solution to differential equations. In these future cases, the function $\mathcal{R}(x)$ is the residual of our differential equation and the domain of the orthogonality is the domain of the differential equation.

1.9 Useful Functional Spaces

So far, we have discussed functional spaces that are constant, odd functions and even functions. We will now expand our concepts into new functional spaces. For example, we define a space of all functions in the domain $0 \leq x \leq 1$ that have a value of zero at both $x = 0$ and $x = 1$. We need to construct a functional set that spans this new space. One possible function set spanning this space is,

$$\phi_i(x) = (x-1)\,x^i \tag{1.146}$$

Since each of our functions obey the end conditions,

$$\phi_i(x=0) = \phi_i(x=1) = 0 \tag{1.147}$$

Any function written as a linear combination of these basis functions would also obey the end conditions,

$$f(x) = \sum_{i=1}^{\infty} a_i \phi_i(x) = \sum_{i=1}^{\infty} a_i (x-1)\,x^i \tag{1.148}$$

This end condition may be written mathematically as:

$$\begin{aligned} f(x=0) &= \sum_{i=1}^{\infty} a_i \underbrace{\phi_i(x=0)}_{=0} = 0 \\ f(x=1) &= \sum_{i=1}^{\infty} a_i \underbrace{\phi_i(x=1)}_{=0} = 0 \end{aligned} \tag{1.149}$$

Another possible functional space is, all functions in the domain $1 \leq x \leq 2$ with the following end condition,

$$\begin{aligned} f(x=1) &= 0 \\ \left.\frac{df}{dx}\right|_{x=2} &= 0 \end{aligned} \tag{1.150}$$

This functional space is a bit challenging to construct a spanning set. However, through use of trigonometry functions and domain mapping a spanning set of functions may be easily generated. If we inspect a simple plot of $\cos(x)$ we notice there always exist a zero derivative at the origin, figure 1.15.

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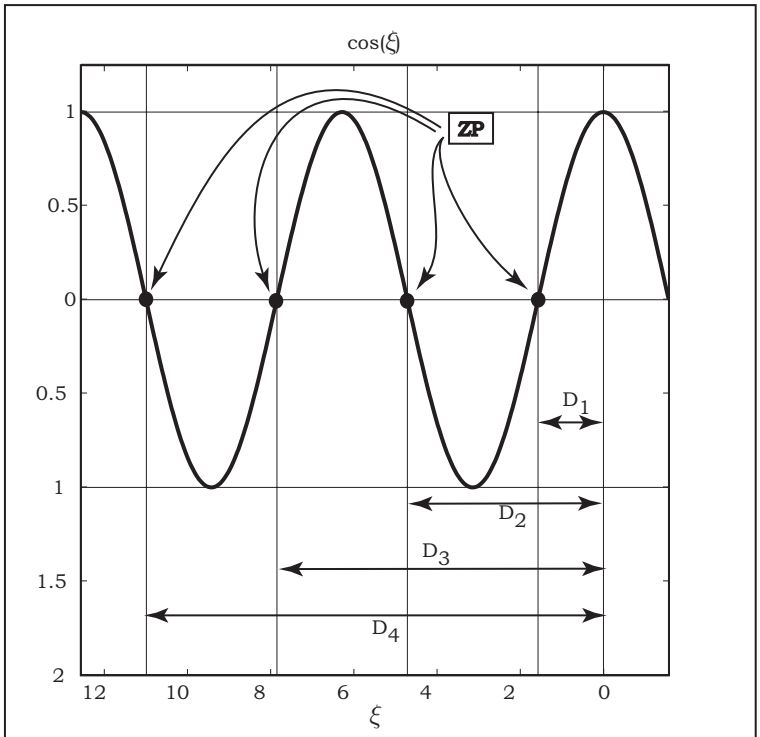


Figure 1.15: Cos function with annotations

In this figure, we notice four different domains. These domains are labeled, D_1, D_2, D_3, and D_4. At the right of each of these domains, $x = 0$, the slope of the cosine function is zero. At the left side of each domain, the function values are zero, indicated by the label ZP. If we could map each of these domains to our $1 \leq x \leq 2$ domain every cosine functions would meet the end conditions and be suitable for a spanning set of functions. First we attempt to map generic domains. The two domains are shown in figure 1.16 below.

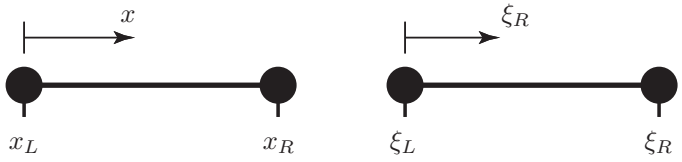


Figure 1.16: Domain Mapping

After careful observation a linear mapping equation can be determined as,

$$\left(\frac{x - x_L}{x_R - x_L}\right) = \left(\frac{\xi - \xi_L}{\xi_R - \xi_L}\right) \tag{1.151}$$

If we insert $\xi = \xi_L$, we obtain $x = x_L$ and if we insert $\xi = \xi_R$, we obtain $x = x_R$. We notice our values of ξ_L and ξ_R from the cosine plot in figure 1.15. We observe that for every domain

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the right most domain point is $\xi_R = 0$. However, the left most domain point, indicated by the ZP in the figure, is related to the left end of domain, D_k. A relationship may be established between domain k and left end point as,

$$\xi_L = -\frac{(2k-1)\,\pi}{2} \tag{1.152}$$

Where, k is the domain number. In addition our domain limits are $1 \leq x \leq 2$. Hence $x_L = 1$ and $x_R = 2$.

Inserting x_L and, x_R, ξ_L and ξ_R into equation 1.151 our mapping equation becomes,

$$(x-1) = \left(\frac{\xi + \frac{(2k-1)\pi}{2}}{\frac{(2k-1)\pi}{2}}\right) = \left(\frac{2\xi}{(2k-1)\,\pi}\right) + 1 \tag{1.153}$$

Rearranging we obtain,

$$\xi = (x-2)\,\frac{(2k-1)\,\pi}{2} \tag{1.154}$$

From this relationship we may now construct our spanning function set, shown below.

$$\phi_k(x) = \cos\left((x-2)\,\frac{(2k-1)\,\pi}{2}\right) \tag{1.155}$$

A plot of several of these function is shown below.

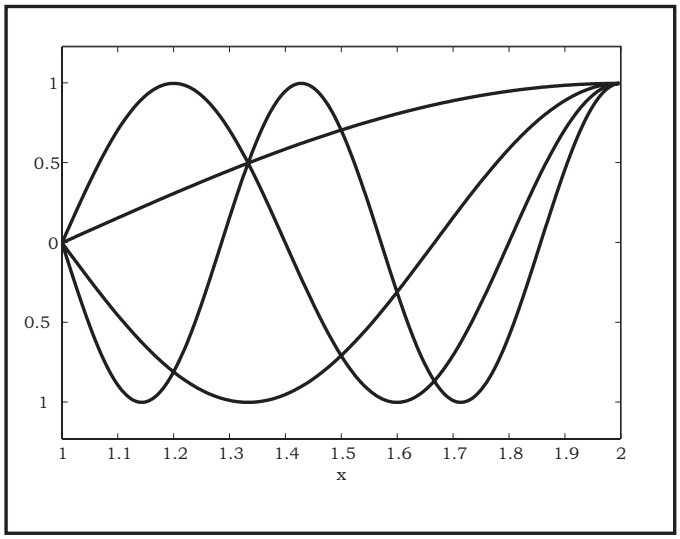


Figure 1.17: Spanning Functions

By observation, each of these functions obey the end conditions in equation 1.150. Therefore, a linear combination of these functions will obey the same end conditions. Any function in this space may be written as,

$$f(x) = \sum_{k=1}^{\infty} a_k \phi_k(x) = \sum_{k=1}^{\infty} a_k \cos\left((x-2)\,\frac{(2k-1)\,\pi}{2}\right) \tag{1.156}$$

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We will use this linear domain mapping in the next chapter to deal with different domains as well as end conditions. With this mapping method as trigonometry we have a powerful tool to match almost any end conditions.

1.10 Functionals, What are they?

We have been using the term functional considerably throughout this chapter. It seems a semi-formal definition of a functional is in order at this point. Since our first calculus course we have been dealing with functionals. By integrating a function over a domain and finding the area under that function, we have been dealing with functionals. For example,

$$\mathcal{A}(f) = \int_{x_l}^{x_r} f(x)\, dx \tag{1.157}$$

is the area under the curve $f(x)$ between x_l and x_r. The notation, $\mathcal{A}(f)$ implies the area is a function of the curve $f(x)$ we place in the integral. We can think of this functional as a mapping of a function, $f(x)$, to a scalar, in this case the area. In effect the area functional listed here is a *function of functions*. The area integral is a linear form since there exist only one function, $f(x)$, in the dependency list and a constant times that function returns that same constant time the original area:

$$\mathcal{A}(c_1 f + c_2 g) = c_1 \mathcal{A}(f) + c_2 \mathcal{A}(g) \tag{1.158}$$

Where f and g are functions of x and c_1 and c_2 are constants. The area functional is indeed a linear functional, see the equation below.

$$\begin{aligned} \mathcal{A}(c_1 f + c_2 g) =& \int_{x_l}^{x_R} [c_1 f(x) + c_2 g(x)]\, dx \\ =& c_1 \int_{x_l}^{x_R} f(x)\, dx + c_2 \int_{x_l}^{x_R} g(x)\, dx \\ \mathcal{A}(c_1 f + c_2 g) =& c_1 \mathcal{A}(f) + c_2 \mathcal{A}(g) \end{aligned} \tag{1.159}$$

One interesting point to notice is the functional dependency. The original function $f(x)$ is a function of x. However, our functional, $\mathcal{A}(f)$, is not a function of x. Our functional, $\mathcal{A}(f)$, is only a function of the function, $f(x)$, placed in the integral. This make sense if we consider what an integration does. The integration we have performed was with respect to dx. Therefore, the integration could be thought of as removing x from the dependency list.

Bilinear functionals have two functions in the dependency list and are linear in each argument.

$$\mathcal{B}(f, g) = \int_{x_l}^{x_R} f(x)\, g(x)\, dx \tag{1.160}$$

This bilinear functional appeared as the first equation of this chapter and is used as an orthogonality statement. A bilinear functional $\mathcal{B}(f, g)$ is linear in each argument if the following is true.

$$\mathcal{B}(c_1 f + c_2 g, r) = c_1 \mathcal{B}(f, r) + c_2 \mathcal{B}(g, r) \tag{1.161}$$

And

$$\mathcal{B}(r, c_1 f + c_2 g) = c_1 \mathcal{B}(r, f) + c_2 \mathcal{B}(r, g) \tag{1.162}$$

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Where r, f and g are functions of x and c_1 and c_2 are constants. We can immediately see our orthogonality functional is, in fact, a bilinear functional.

$$\begin{aligned}\mathcal{B}(c_1 f + c_2 g, r) =& \int_{x_l}^{x_R} [c_1 f(x) + c_2 g(x)] r(x)\, dx \\ =& c_1 \int_{x_l}^{x_R} f(x) r(x)\, dx + c_2 \int_{x_l}^{x_R} g(x) r(x)\, dx \\ =& c_1 \mathcal{B}(f, r) + c_2 \mathcal{B}(g, r)\end{aligned} \tag{1.163}$$

$$\begin{aligned}\mathcal{B}(r, c_1 f + c_2 g) =& \int_{x_l}^{x_R} r(x) [c_1 f(x) + c_2 g(x)]\, dx \\ =& c_1 \int_{x_l}^{x_R} r(x) f(x)\, dx + c_2 \int_{x_l}^{x_R} r(x) g(x)\, dx \\ =& c_1 \mathcal{B}(r, f) + c_2 \mathcal{B}(r, g)\end{aligned} \tag{1.164}$$

One may ask at this point what are functionals that are not linear. On typical example is,

$$\mathcal{Q}(f) = \int_{x_l}^{x_R} [f(x)]^2\, dx \tag{1.165}$$

If we test this functional we note its nonlinear characteristic.

$$\mathcal{Q}(c_1 f) = [c_1]^2 \int_{x_l}^{x_R} [f(x)]^2\, dx \neq c_1 \mathcal{Q}(f) \tag{1.166}$$

In fact, this functional is call the quadratic functional. This is primarily due to it quadratic characteristic. The relationship between a quadratic functional and the bilinear form is apparent by writing this equation as;

$$\mathcal{Q}(f) = \mathcal{B}(f, f) = \int_{x_l}^{x_R} [f(x)]^2\, dx \tag{1.167}$$

In the next chapter we shall see how the weighted residual statement may be transformed into a quadratic functional. In addition, we shall see how a symmetric bilinear form will always result in a quadratic functional who's minimum is the solution to our governing equation. We will expand on some of these functional concepts in the next chapter. As it turns out, there is a close connection between functional analysis, variational methods and real analysis. We shall investigate some of these connections when we begin to look at the weighted residual methods.

1.11 Functional to Matrix Mappings

Linear and bilinear functionals which operate on spanning function sets produce something very interesting and useful, as we shall see later. Remember a spanning set of functions are a group of functions which can be used to represent any function in a particular functional space. These spanning functions can be thought of as a basis for that functional space. In the *first fundamental lemma of variational calculus* we forced a function, $\mathcal{R}(x)$, to be orthogonal

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to an infinite set of linearly independent functions. In fact, this is the extreme case. We can force a function our to be orthogonal to a finite set of linearly independent functions. For example, let assume we have a function $\mathcal{R}(x)$ as shown below.

$$\mathcal{R}(x) = \sum_{i=1}^{5} a_i \psi_i(x) \tag{1.168}$$

Examining the orthogonality statement we only need to make our function orthogonal to 5 linearly independent functions, $\psi_i(x)$'s.

$$\int_{x_L}^{x_R} \mathcal{R}(x)\,\psi_j(x)\,dx \quad \text{for} \quad j = 1,2,3,4,5 \tag{1.169}$$

If we insert our function from equation 1.168 into equation 1.169 we obtain the following.

$$\sum_{i=1}^{5} \left[\int_{x_L}^{x_R} \psi_i(x)\,\psi_j(x)\,dx \right] a_i \quad \text{for} \quad j = 1,2,3,4,5 \tag{1.170}$$

The term in square brackets is a bilinear functional as well as a matrix,

$$M_{ij} = \mathcal{B}(\psi_i, \psi_j) = \int_{x_L}^{x_R} \psi_i(x)\,\psi_j(x)\,dx \tag{1.171}$$

In our case the matrix is a 5 by 5 and can be written in a matrix equation as,

$$\begin{bmatrix} \mathcal{B}(\psi_1,\psi_1) & \mathcal{B}(\psi_1,\psi_2) & \mathcal{B}(\psi_1,\psi_3) & \mathcal{B}(\psi_1,\psi_4) & \mathcal{B}(\psi_1,\psi_5) \\ \mathcal{B}(\psi_2,\psi_1) & \mathcal{B}(\psi_2,\psi_2) & \mathcal{B}(\psi_2,\psi_3) & \mathcal{B}(\psi_2,\psi_4) & \mathcal{B}(\psi_2,\psi_5) \\ \mathcal{B}(\psi_3,\psi_1) & \mathcal{B}(\psi_3,\psi_2) & \mathcal{B}(\psi_3,\psi_3) & \mathcal{B}(\psi_3,\psi_4) & \mathcal{B}(\psi_3,\psi_5) \\ \mathcal{B}(\psi_4,\psi_1) & \mathcal{B}(\psi_4,\psi_2) & \mathcal{B}(\psi_4,\psi_3) & \mathcal{B}(\psi_4,\psi_4) & \mathcal{B}(\psi_4,\psi_5) \\ \mathcal{B}(\psi_5,\psi_1) & \mathcal{B}(\psi_5,\psi_2) & \mathcal{B}(\psi_5,\psi_3) & \mathcal{B}(\psi_5,\psi_4) & \mathcal{B}(\psi_5,\psi_5) \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{Bmatrix} \tag{1.172}$$

Therefore, a bilinear functional will yield a matrix while a linear functional will yield a vector. If we use a set of orthogonal basis functions to generate this matrix our off diagonal terms we be zero, $\mathcal{B}(\psi_i, \psi_j) = 0 \quad \forall \; i \neq j$. For this orthogonal case our matrix would become,

$$\begin{bmatrix} \mathcal{B}(\psi_1,\psi_1) & 0 & 0 & 0 & 0 \\ 0 & \mathcal{B}(\psi_2,\psi_2) & 0 & 0 & 0 \\ 0 & 0 & \mathcal{B}(\psi_3,\psi_3) & 0 & 0 \\ 0 & 0 & 0 & \mathcal{B}(\psi_4,\psi_4) & 0 \\ 0 & 0 & 0 & 0 & \mathcal{B}(\psi_5,\psi_5) \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{Bmatrix} \tag{1.173}$$

However, what if we did not begin with an orthogonal set of functions? How could we obtain a diagonal matrix at this point? Is it too late to force our matrix into a diagonal state? The answer to these questions is, we can determine a set of orthonormal functions from the matrix. First we must realize, any diagonal matrix obtained from the equation 1.171 is done so from orthogonal functions. The reverse is also true if we forced our matrix to be diagonal in a mathematically legal sense, then the orthogonal functions may be determined from this matrix. The question now is how? To answer this question we turn to eigenvalue-eigenvector analysis of matrices.

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We begin with the standard matrix eigenvalue problem. Given a matrix $[M]$ we wish to determine a vector, $\{t\}$ and a scalar, λ, such that the following is true,

$$[M]\{t\}_i = \lambda_i \{t\}_i \tag{1.174}$$

In fact, for an $N \times N$ matrix, $[M]$, there exist N such possibilities. Hence the index i on each term. If we wrote this equation in full matrix form the eigenvalue-eigenvector statement would read,

$$[M][T] = [T][\Lambda] \tag{1.175}$$

Where, $[T]$, is a matrix who's columns are eigenvectors and, $[\Lambda]$ is a diagonal matrix who's diagonals are the eigenvalues. This equation may be premultiplied by the inverse of the eigenvector matrix to obtain,

$$[T]^{-1}[M][T] = [\Lambda] \tag{1.176}$$

As it turns out, if each eigenvector's norm is unity the eigenvector matrix's inverse and transpose are equal,

$$[T]^{T}[M][T] = [\Lambda] \tag{1.177}$$

As it turns out this scaling is exactly what the programmers of Matlab has decided upon. Proof of their efforts may be seen in the following Matlab script,

```
 1 %% Eigen Test
 2 % Dr. Fithen
 3 M=[1 4 5 0 0;
 4    4 7 9 1 0;
 5    5 9 4 8 9;
 6    0 1 8 3 0;
 7    0 0 9 0 8]
 8 [T,L]=eig(M)
 9 % Look @ scale of each eigenvector
10 for i=1:5
11     Norm=T(:,i)'*T(:,i);
12     fprintf(' Norm of eigenvector %i is ==%f\n',i,Norm);
13 end
14 % Transpose==Inverse???
15 T'-inv(T)
16 % M==T'*M*T ????
17 T'*M*T
```

Figure 1.18: Eigen Matlab Listing

Any symmetric matrix may be placed in this routine, lines 3 to 7, and the results will be the same. The matlab routine `eig` will return two matrices, `T` is our transformation matrix, line 8 and $[T]$ in equations 1.175 - 1.177, and the matrix `L` in line 8 of the Matlab script is the same as our eigenvalue matrix $[\Lambda]$. From lines 10 to 13 of our Matlab script that indeed the programmers of Matlab have chosen to scale our eigenvectors to unity. As a result, the transformation matrix, $[T]$, inverse and transpose is equal, line 15 of the Matlab script. Finally, line 17 of the Matlab script proves equation 1.177.

Now that we have determined a way of making a matrix diagonal through a transformation $[T]$, how do we obtain our basis functions from this transformation? From a previous section, beginning on page 24, we became familiar with the concepts of polynomial transformations.

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We may write a polynomial transformation for our basis functions as,

$$\psi_i(x) = \sum_{k=1}^{5} T_{ik}\varphi_k(x) \tag{1.178}$$

Where $\psi_i(x)$ is a set of non-orthogonal basis functions, equation 1.168. $\varphi_k(x)$ is a set of orthogonal basis functions, and T_{ik} is a transformation matrix between the two sets of basis functions. If we insert this relationship into our bilinear form we obtain,

$$\mathcal{B}(\psi_i, \psi_j) = \mathcal{B}\left(\sum_{k=1}^{5} T_{ik}\varphi_k, \sum_{l=1}^{5} T_{jl}\varphi_l\right) \tag{1.179}$$

Since this equation is based on a bilinear form and the transformation matrices are constants, this equation becomes,

$$\mathcal{B}(\psi_i, \psi_j) = \sum_{k=1}^{5}\sum_{l=1}^{5} T_{ik}\mathcal{B}(\varphi_k, \varphi_l)T_{jl} \tag{1.180}$$

Since the basis functions $\varphi_k(x)$ are orthogonal, the term $\mathcal{B}(\varphi_k, \varphi_l)$ is a diagonal matrix. This matrix is in fact the $[\Lambda]$ term in equation 1.177. Likewise the bilinear form $\mathcal{B}(\psi_i, \psi_j)$ is the same as the $[M]$ matrix in equation 1.177. As a result, equation 1.180 may be written as,

$$M_{ij} = \sum_{k=1}^{5}\sum_{l=1}^{5} T_{ik}\Lambda_{kl}T_{jl} \tag{1.181}$$

The two summations in this equation represent matrix multiplications. However, the summation on l could be written correct if we transposed the last matrix,

$$M_{ij} = \sum_{k=1}^{5}\sum_{l=1}^{5} T_{ik}\Lambda_{kl}T^T_{lj} \tag{1.182}$$

With this alteration we may immediately case this equation into matrix notation,

$$[M] = [T][\Lambda][T]^T \tag{1.183}$$

Comparing this equation with equations 1.176 and 1.177 we notice the significance of the transformation matrix generated from an eigenvalue/vector analysis. This transformation matrix is in fact the transformation matrix between the two polynomial basis functions, one non-orthogonal and one orthogonal.

Example

Given a basis function set shown below,

$$\begin{aligned}\psi_1(x) &= 1\\ \psi_2(x) &= x\\ \psi_3(x) &= x^2\\ \psi_4(x) &= x^3\\ \psi_5(x) &= x^4\end{aligned} \tag{1.184}$$

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We wish to generate a orthogonal basis function via eigenvalue/vector analysis. We begin by constructing our $[M]$ matrix by using the bilinear form,

$$M_{ij} = \mathcal{B}(\psi_i, \psi_j) = \int_{x_l}^{x_r} \psi_i(x)\,\psi_j(x)\,dx \tag{1.185}$$

We shall select our domain as $0 \leq x \leq 1$.
After integrating our terms we obtain the matrix,

$$[M] = \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 & 1/5 \\ 1/2 & 1/3 & 1/4 & 1/5 & 1/6 \\ 1/3 & 1/4 & 1/5 & 1/6 & 1/7 \\ 1/4 & 1/5 & 1/6 & 1/7 & 1/8 \\ 1/5 & 1/6 & 1/7 & 1/8 & 1/9 \end{bmatrix} \tag{1.186}$$

Using Matlab's eig function we obtain the transformation matrix,

$$[T] = \begin{bmatrix} -0.0062 & 0.0472 & 0.2142 & -0.6019 & 0.7679 \\ 0.1167 & -0.4327 & -0.7241 & 0.2759 & 0.4458 \\ -0.5062 & 0.6674 & -0.1205 & 0.4249 & 0.3216 \\ 0.7672 & 0.2330 & 0.3096 & 0.4439 & 0.2534 \\ -0.3762 & -0.5576 & 0.5652 & 0.4290 & 0.2098 \end{bmatrix} \tag{1.187}$$

As as result the orthogonal functions become,

$$\begin{Bmatrix} \varphi_1(x) \\ \varphi_2(x) \\ \varphi_3(x) \\ \varphi_4(x) \\ \varphi_5(x) \end{Bmatrix} = \begin{bmatrix} -0.0062 & 0.1167 & -0.5062 & 0.7672 & -0.3762 \\ 0.0472 & -0.4327 & 0.6674 & 0.2330 & -0.5576 \\ 0.2142 & -0.7241 & -0.1205 & 0.3096 & 0.5652 \\ -0.6019 & 0.2759 & 0.4249 & 0.4439 & 0.4290 \\ 0.7679 & 0.4458 & 0.3216 & 0.2534 & 0.2098 \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \end{Bmatrix} \tag{1.188}$$

Where the matrix form of equation 1.178 was used,

$$\{\varphi\} = [T]^{-1}\{\psi\} = [T]^T\{\psi\} \tag{1.189}$$

One very important property should be noted at this point, every polynomial function in the orthogonal set, $\varphi_i(x)$, is fourth order. This may be seen by multiplying out equation 1.188 as well as by inspecting the transformation matrix. Compare this result to that of equation 1.134 and 1.135 where the polynomial order of term i is i^{th} order. As a result the transformation matrix shown in equation 1.135 contains zeros above the diagonal. This same feature exist in Legendre and Tchebychev polynomials, equation 1.86 and 1.87. As a result, the eigenvalue/vector construction of orthogonal functions will not generate hierarchial type functions. Any function who's polynomial order of term i is i^{th} order can be used as hierarchial functions. This property may be easily observed by inspecting the transformation matrix between the basis functions in question and monomial basis functions. We shall use these hierarchial basis functions extensively in the finite element method.

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A complete listing of the Matlab script which produces the orthogonal basis function is shown in figure **??**.

```
1 %% Orthogonal Construction via Eigen
2 % Dr. Fithen
3 clear
4 syms x
5 % Define our monomials
6 psi=[1;x;x^2;x^3;x^4]
7 for i=1:5
8     for j=1:5
9         M(i,j)=int(psi(i)*psi(j),0,1)
10    end
11 end
12 % convert symbolic to double
13 M=subs(M)
14 %Call Matlab's eig function
15 %to determine eignevalus & vectors
16 [T,L]=eig(M)
17 %Use transformation matrix to obtain new
18 %orthogonal basis functions, varphi
19 varphi=T'*psi
20 for i=1:5
21     for j=1:5
22         Mnew(i,j)=int(varphi(i)*varphi(j),0,1);
23     end
24 end
25 Mnew=subs(Mnew)
```

Figure 1.19: Eigen Construction Matlab Script

In this script, line 6 defines our monomials. This line may be changed to suit the given polynomial functions. Lines 7 to 11 produce our mass matrix from our bilinear form. Line 13 simply converts this matrix from a symbolic form into double precision. Line 16 calls Matlab's internal eigenvalue/vector routine. This routine returns the transformation matrix as `T`, and the diagonal matrix as `L`. The transformation matrix, `T`, is a matrix who's columns are the eigenvectors of matrix `M`. While the matrix, `L`, contains the eigenvalues of `M` on its diagonals. The new orthogonal basis functions are obtained in line 19. line 20 through 25 calculate the new mass matrix, which is diagonal. In fact this matrix is exactly the matrix returns as `L` on line 16.

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1.12 Problems

Problem 1

Given the following function, determine and plot the 5, 10, 20, 40 and 100 term (truncated) Fourier series expansion.

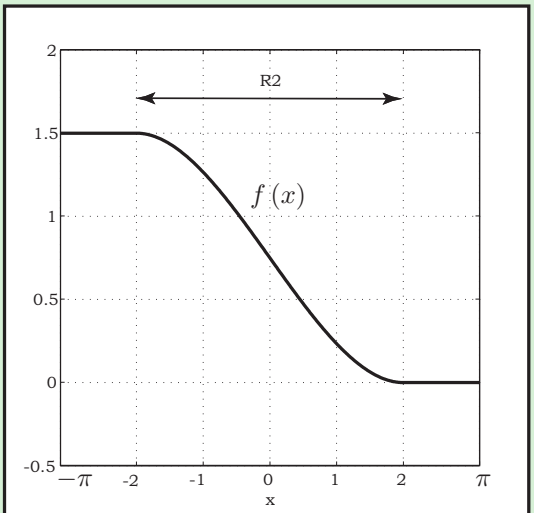


Figure 1.20: Problem 1

The function may be written as,

$$f(x) = \begin{cases} 1.5 & \text{for} \quad -\pi \leq x \leq -2 \\ \frac{3}{4} - \frac{9}{16}\,x + \frac{3}{64}\,x^3 & \text{for} \quad -2 \leq x \leq +2 \\ 0 & \text{for} \quad +2 \leq x \leq \pi \end{cases}$$

Problem 2

Given the following function, determine and plot the 10, 50, 100, 200 and 1000 term (truncated) Fourier series expansion .

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Problem 2: Cont

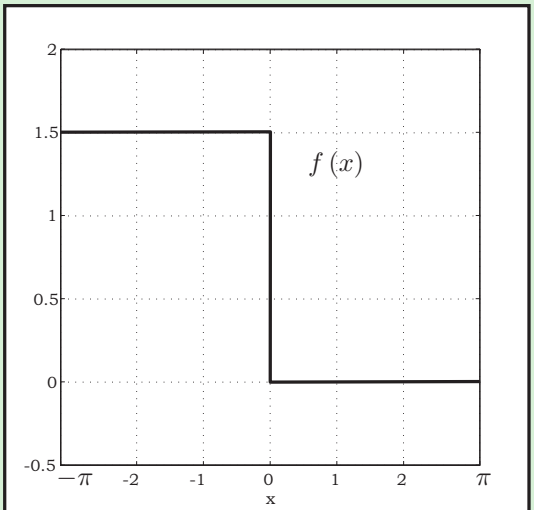


Figure 1.21: Problem 2: Step Function

Notice that this function is discontinuous at $x = 0$. Explain how this discontinuous function be represented by a set of functions that is continuous?

Problem 3

Given the following function, determine and plot the 10, 20, 50, 200 and 500 term (truncated) Fourier series expansion.

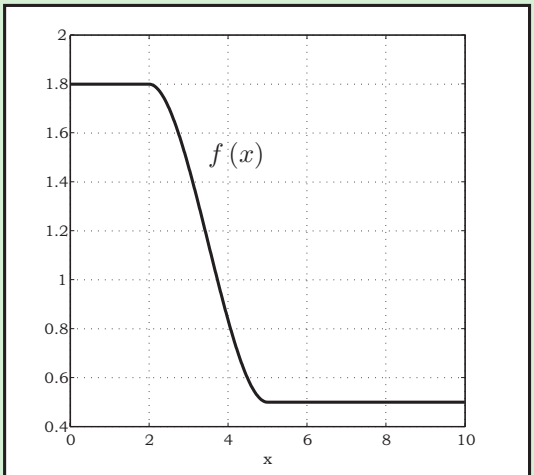


Figure 1.22: Problem 3

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Problem 3: Cont

The function may be written as,

$$f(x) = \begin{cases} 1.0 & \text{for} \quad 0 \le x \le 2 \\ -\frac{19}{27} - \frac{91}{90}\,x^2 + \frac{26}{9}\,x + \frac{13}{135}\,x^3 & \text{for} \quad 2 \le x \le 5 \\ 0.5 & \text{for} \quad 5 \le x \le 10 \end{cases}$$

NOTE: Use the transformation, equation 1.151, to transform the function back to the Fourier space, $-\pi \le x \le \pi$, to perform the integrations.

Problem 4

Given the following function, determine and plot the 2, 5, 10, 15, and 20 term (truncated) Legendre series expansion.

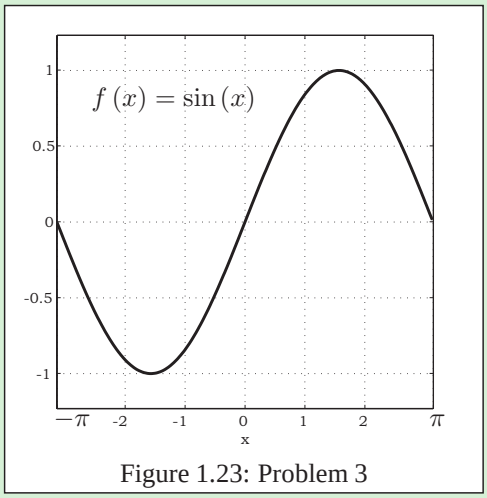


Figure 1.23: Problem 3

NOTE: Use the transformation, equation 1.151, to transform the function back to the Legendre space, $-1 \le x \le 1$, to perform the integrations.
Use the transformation matrix between Legendre and monomial polynomials to prove the resulting polynomials give the Taylor series expansion of the sin function, equation 1.55.

Problem 5

Given the following function, determine and plot the 2, 5, 10, 15, and 20 term (truncated) Tchebychev series expansion.

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Problem 5: Cont

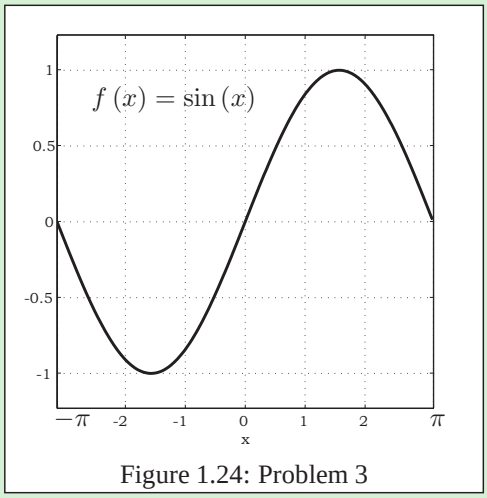


Figure 1.24: Problem 3

NOTE: Use the transformation, equation 1.151, to transform the function back to the Tchebychev space, $-1 \leq x \leq 1$, to perform the integrations.

Problem 6

Develop a transformation matrix between Legendre and Tchebychev polynomials. Test this transformation matrix by proving the results from Problem 4 will transform to the results of problem 5.

Problem 7

Develop a transformation matrix between Legendre polynomials and the orthonormal polynomials, equation 1.134 up to tenth order. Note: More terms will need to be developed, figure 1.14.

$$\{\psi\} = \left[T^{\psi\mathcal{P}}\right]\{\mathcal{P}\}$$

Prove that this transformation is the same as,

$$\{\psi\} = \left[T^{\psi\mathcal{P}}\right]\{\mathcal{P}\} = \left[T^{\psi\mathcal{P}}\right]\left[T^{\mathcal{P}\mathcal{T}}\right]\{\mathcal{T}\} = \left[T^{\psi\mathcal{P}}\right]\left[T^{\mathcal{P}\mathcal{T}}\right]\left[T^{\mathcal{T}\mathcal{M}}\right]\{\mathcal{M}\}$$

Problem 8

Determine the Legendre series expansion of the function given in problem 1. Plot a truncated series using 5, 10, 20 and 50 terms.

Problem 9

Determine the Tchebychev series expansion of the function given in problem 1. Plot a truncated series using 5, 10, 20 and 50 terms.

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Problem 10

Determine the Orthonormal series expansion, equation 1.134, of the function given in problem 1. Plot a truncated series using 5, 10, 20 and 50 terms. Note: More terms will need to be developed, figure 1.14.

Problem 11

Develop a new set of functions, shown below.

$$\begin{aligned}\chi_3(x) &= \mathcal{P}_2(x) - \mathcal{P}_0(x)\\ \chi_4(x) &= \mathcal{P}_3(x) - \mathcal{P}_1(x)\\ \chi_5(x) &= \mathcal{P}_4(x) - \mathcal{P}_2(x)\\ \chi_6(x) &= \mathcal{P}_5(x) - \mathcal{P}_3(x)\\ \chi_7(x) &= \mathcal{P}_6(x) - \mathcal{P}_4(x)\\ \chi_8(x) &= \mathcal{P}_7(x) - \mathcal{P}_5(x)\end{aligned}$$

- Plot these six functions in the domain, $-1 \leq x \leq +1$. Comment on the end conditions.
- Develop the orthogonal (mass) matrix, equation 1.172.
- From the orthogonality (mass) matrix are these functions orthogonal?
- Use the Gram-Schmidt to develop a set of orthonormal functions from these six functions.

Problem 12

Develop a new set of functions, shown below.

$$\begin{aligned}\chi_3(x) &= \mathcal{T}_2(x) - \mathcal{T}_0(x)\\ \chi_4(x) &= \mathcal{T}_3(x) - \mathcal{T}_1(x)\\ \chi_5(x) &= \mathcal{T}_4(x) - \mathcal{T}_2(x)\\ \chi_6(x) &= \mathcal{T}_5(x) - \mathcal{T}_3(x)\\ \chi_7(x) &= \mathcal{T}_6(x) - \mathcal{T}_4(x)\\ \chi_8(x) &= \mathcal{T}_7(x) - \mathcal{T}_5(x)\end{aligned}$$

- Plot these six functions in the domain, $-1 \leq x \leq +1$. Comment on the end conditions.
- Develop the orthogonal (mass) matrix, equation 1.172.
- From the orthogonality (mass) matrix are these functions orthogonal?
- Use the Gram-Schmidt to develop a set of orthonormal functions from these six functions.

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Problem 13

Use the set of functions shown below and do the following.

$$\chi_i(x) = x(1-x)^i$$

- Plot these six functions in the domain, $-1 \leq x \leq +1$. Comment on the end conditions.
- Develop the orthogonal (mass) matrix, equation 1.172.
- From the orthogonality (mass) matrix are these functions orthogonal?
- Use the Gram-Schmidt to develop a set of orthonormal functions from these six functions.

Problem 14

Work the last three problems and compare the orthonormal functions generated from the Gram-Schmidt process. Are they the same functions or different? Why?

Problem 15

Which of the following function sets are complete? Why or why not?

A:

$$\begin{aligned}\phi_1(x) &= x\\ \phi_2(x) &= x^2\\ \phi_3(x) &= x^3\\ \phi_4(x) &= x^4\\ \phi_5(x) &= x^5\\ \phi_6(x) &= x^6\\ \phi_7(x) &= x^7\end{aligned}$$

B:

$$\begin{aligned}\phi_1(x) &= \cos(x)\\ \phi_2(x) &= x^2\\ \phi_3(x) &= x^3\\ \phi_4(x) &= x^4\\ \phi_5(x) &= x^5\\ \phi_6(x) &= x^6\\ \phi_7(x) &= x^7\end{aligned}$$

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Problem 15: Cont

C:

$$\begin{aligned}\phi_1(x) &= \cos(x)\\ \phi_2(x) &= x\\ \phi_3(x) &= x^2\\ \phi_4(x) &= x^3\\ \phi_5(x) &= x^4\\ \phi_6(x) &= x^5\\ \phi_7(x) &= x^6\end{aligned}$$

D:

$$\begin{aligned}\phi_1(x) &= \cos(x)\\ \phi_2(x) &= x\\ \phi_3(x) &= x^2\\ \phi_4(x) &= x^3\\ \phi_5(x) &= x^4\\ \phi_6(x) &= x^5\\ \phi_7(x) &= x^6\end{aligned}$$

E:

$$\begin{aligned}\phi_1(x) &= 1\\ \phi_2(x) &= x\\ \phi_3(x) &= x^2\\ \phi_4(x) &= x^3\\ \phi_5(x) &= x^4\\ \phi_6(x) &= x^5\\ \phi_7(x) &= x^6\end{aligned}$$

F:

$$\begin{aligned}\phi_1(x) &= 1\\ \phi_2(x) &= 3x+7\\ \phi_3(x) &= 3x+3x^2+1\\ \phi_4(x) &= 7x^2+10x^3\\ \phi_5(x) &= 5x^4\\ \phi_6(x) &= 7x^2+3x^3+2\\ \phi_7(x) &= x^6\end{aligned}$$

G:

$$\begin{aligned}\phi_1(x) &= 1+x\\ \phi_2(x) &= x+x^2\\ \phi_3(x) &= x^2+x^3\\ \phi_4(x) &= x^3+x^4\\ \phi_5(x) &= x^4+x^5\\ \phi_6(x) &= x^5+x^6\\ \phi_7(x) &= x^6+x^7\end{aligned}$$

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Problem 16

Use the Gram-Schmidt process to produce a set of orthonormal functions from:

$$\begin{aligned}\phi_1(x) &= 1 + x \\ \phi_2(x) &= x + x^2 \\ \phi_3(x) &= x^2 + x^3 \\ \phi_4(x) &= x^3 + x^4 \\ \phi_5(x) &= x^4\end{aligned}$$

Problem 17

Which of the following are bilinear or linear functionals?

A $$\mathcal{B}(f,g) = \int_0^5 \frac{\partial f}{\partial x}\, g\, dx$$

B $$\mathcal{B}(f,g) = \int_0^1 f \frac{\partial f}{\partial x}\, g\, dx$$

C $$\mathcal{B}(f,g) = \int_0^1 \frac{\partial f}{\partial x}\, \frac{\partial g}{\partial x}\, dx$$

D $$\mathcal{B}(f,g) = \int_0^1 \frac{\partial^2 f}{\partial x^2}\, \frac{\partial^2 g}{\partial x^2}\, dx$$

E $$\mathcal{Q}(f) = \int_0^1 \frac{\partial^2 f}{\partial x^2}\, dx$$

F $$\mathcal{Q}(f) = \int_0^1 \frac{\partial f}{\partial x}\, dx$$

G $$\mathcal{Q}(f) = \int_0^1 \left[\frac{\partial f}{\partial x}\right]^2 dx$$

H $$\mathcal{Q}(f) = \int_0^1 f\, dx$$

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Problem 18

Use the Gram-Schmidt process to produce a set of ten orthonormal functions from the function set shown in equation 1.155. Note: The domain must be, $1 \leq x \leq 2$.

$$\phi_k(x) = \cos\left((x-2)\frac{(2k-1)\pi}{2}\right)$$

Problem 19

Using the concepts from figure 1.16 and equation 1.151 to develop a set of functions with the following end conditions:

A
$$\begin{aligned} \left.\frac{df}{dx}\right|_{x=2} &= 0 \\ f(x=4) &= 0 \end{aligned}$$

B
$$\begin{aligned} f(x=9) &= 0 \\ \left.\frac{df}{dx}\right|_{x=20} &= 0 \end{aligned}$$

C
$$\begin{aligned} \left.\frac{df}{dx}\right|_{x=10} &= 0 \\ \left.\frac{df}{dx}\right|_{x=20} &= 0 \end{aligned}$$

Use the Gram-Schmidt method to convert these function sets to an orthonormal set of functions.

Problem 20

Use the functions defined in problems 10 and 11 above and develop a new set of functions using *an altered* Gram-Schmidt orthonormalization process. This *altered* Gram-Schmidt orthonormalization process uses the following bilinear form:

$$\mathcal{B}(\phi_i, \phi_j) = \int_{-1}^{+1} \frac{d\phi_i}{dx}\frac{d\phi_j}{dx}dx$$

Plot and compare these functions to those orthonormal functions generated in problems 10 and 11.
Find the matrix K_{ij}, see below, for the functions defined in problems 10 and 11. In addition, find the matrix K_{ij} for our Gram-Schmidt generated orthonormal functions.

$$K_{ij} = \mathcal{B}(\phi_i, \phi_j) = \int_{-1}^{+1} \frac{d\phi_i}{dx}\frac{d\phi_j}{dx}dx$$

NOTE: This matrix will be a 5x5 since only 5 terms are included in problems 10 and 11.

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Problem 21

Using the Gram-Schmidt process definition, and the concepts from the definition of bilinear forms prove the following.

$$\mathcal{B}(\psi_1, \psi_2) = 0$$

Where, repeating the first three lines of the Gram-Schmidt process we have.

$$\psi_1 = \frac{\phi_1}{\sqrt{\mathcal{B}(\phi_1, \phi_1)}}$$

$$\widetilde{\psi}_2 = \phi_2 - \psi_1 \mathcal{B}(\psi_1, \phi_2)$$

$$\psi_2 = \frac{\widetilde{\psi}_2}{\sqrt{\mathcal{B}\left(\widetilde{\psi}_2, \widetilde{\psi}_2\right)}}$$

Problem 22

Given a set of functions,

$$\begin{aligned}
\psi_1(x) &= (1-x)\,x \\
\psi_2(x) &= (1-x)\,x^2 \\
\psi_3(x) &= (1-x)\,x^3 \\
\psi_4(x) &= (1-x)\,x^4 \\
\psi_5(x) &= (1-x)\,x^5 \\
\psi_6(x) &= (1-x)\,x^6 \\
\psi_7(x) &= (1-x)\,x^7
\end{aligned}$$

Using the Eigenvalue/vector process, figure 1.19, to determine the mass matrix form it bilinear form. Then determine a set of orthogonal functions using eigenvalue/vector analysis.

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