Introduction to Finite Element Computations for Heat Transfer and Fluid Flow

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Chapter 1

Introduction

The scope of the material contained in this text is very focused. The only problem types considered are steady and unsteady incompressible flows with heat transfer. The heat transfer modes considered are convection and conduction. The modes of convective heat transfer include force and natural convection. By including only this material I am not excluding the importance of other flow types. I am however, focusing on these flow and heat transfer modes in order to treat these situations in a more focused manner. This text is not intended as a complete review of research work in this area. For this reason, several valid approaches are not considered or even mentioned. However, if a reader is interested in understanding and programming the finite element method for incompressible fluid flow and heat transfer this book will guide the reader in that direction. With that in mind maybe the title of text should have been "*How to Program the Finite Element Method for Incompressible Fluid Flow and Heat Transfer*"

1.1 Why Finite Elements?

There are other methods equally as valid that may be used to solve incompressible fluid flow and heat transfer equations. However, finite elements has been growing in popularity over the last few decades. This growth is in part due to the large flexibility of the method. In fact, the same subroutines used in fluid mechanics programs are used in solid mechanics, electromagnetics, heat transfer, as well as many other problems. Another feature of the finite element method is its ability to provide higher order elements with very little and sometime no algorithmic changes inside the computer codes. And finally, the finite element method has gain significant recruits due to the elegant mathematical basis and its elegant error analysis capabilities.

1.2 Prerequisite Material

In order to fully comprehend the material presented in this text it would be helpful to have already been exposed to both fluid mechanics as well as convection and conduction heat transfer.Navier-Stokes equations as well as the heat transfer equations are written in their

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incompressible forms and very little explanation and no derivations of these equations are given.

It would be helpful to have had some exposure to the finite element method. However, a traditional finite element course is not a prerequisite for this text. Only topics directly relating to finite elements in incompressible flows and heat transfer are included. As a result, several areas of the finite element method are never presented. This type material is left to finite element text with a broader focus.

1.3 Why Matlab

Matlab has become popular among a majority of engineering curriculums. This is primarily due to ability to handle complex engineering problems with relative simplicity. More important to us is its ability to handle matrices with ease. On the down side, Matlab is often much slower than a formal programming language like C or Fortran. However, for our purposes Matlab is very valuable as an educational tool. If the goal later becomes the solution of large scale fluid mechanics and heat transfer problems, a formal language like C or Fortran may become necessary.

1.4 Elements used in this text

There are an incredible number of elements and basis functions available for use in the finite element method. In fact, this is still an active area of research in the finite element community. However, in keeping with our focused intent of this text only one type of element will be presented. In one dimension this element will be a linear element. In two dimensions the element type will be a triangular element with straight edges. In three dimensions the element type will be a tetrahedron made up of flat triangular patches. The primary reason for these elements is the nature of the Jacobian of their transformations. As we shall see, all of these elements poses a Jacobian which is constant throughout the element. As a result, numerical integration of these elements are not necessary. In fact, the element matrix only need a scale factor, the Jacobian, from element to element. This will significantly reduce the computational complexity required as we increase the order of our elements.

1.5 Grid generation

With the use of only triangular and tetrahedral elements a grid generator must be obtained that will give such elements. Just such a tool is available for use directly in Matlab. This tool is called **DisMesh** developed by Per-Olof Persson and Gilbert Strang at MIT. [**?**]. This grid generation works very well for both triangular and tetrahedral elements.

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Chapter 2

Governing Equations

As stated previously, this text focuses on one particular flow type and heat transfer type. With this in mind we shall only present governing equations to these types. A full derivation and classification of all flow heat transfer types may be see in the fluid mechanics and heat transfer texts, [**?**], [**?**], [**?**] and [**?**].

2.1 Continuity

The continuity equation for incompressible flows may be thought of as a constraint on the flow field. In these situations the density of the fluid, ρ, does not change with respect to time or physical location. For example, the flow of liquid water, or for that matter, the flow of any liquid may be considered an incompressible flow. In addition, flows of gases at low Mach numbers may also be considered as incompressible flows. In cartesian coordinates the continuity equation may be written as,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{2.1}$$

Where

- u is the velocity in the x direction.
  - u is a function of space and time, $u\left(x, y, z, t\right)$.
- v is the velocity in the y direction.
  - v is a function of space and time, $v\left(x, y, z, t\right)$.
- w is the velocity in the z direction.
  - w is a function of space and time, $w\left(x, y, z, t\right)$.

As a result the velocity may be written in vector form as,

$$\vec{V} = u\vec{e}_x + v\vec{e}_y + w\vec{e}_z = u\hat{\imath} + v\hat{\jmath} + w\hat{k} \tag{2.2}$$

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- $\vec{e}_x = \hat{\imath}$ is the base vector in the x direction.
- $\vec{e}_y = \hat{\jmath}$ is the base vector in the y direction.
- $\vec{e}_z = \hat{k}$ is the base vector in the z direction.

2.2 Linear Momentum Equations

Linear momentum equations result from the application of Newton's laws applied to a volume of fluid. Application of Newton's second law result in three equations, one for each dimension. The resulting equations are called Navier-Stokes equations after Frenchman L. M. Navier (1758-1836) and Englishman Sir G. G. Stokes (1819-1903). These three equations along with the continuity equation represent four equations along with four unknowns three velocities, (u, v, w), and the pressure p. The viscous portion of the equations presented below have been reduced by use of the continuity equation, 2.1.

There are two difficult problems with the solution of these four equations. The first problem results from the nonlinearity in the convective terms. These terms are second, third and fourth terms on the left hand side of each equation. These terms add significant complexity as well as create some very interesting solutions as we shall see later. These terms also are the culprit with respect to the grand challenge problem of turbulent flows. As a result, the unsteady version of these terms have been modeled through statistical methods to obtain a variety of turbulence modeling equations. Also, as a result of these terms, fluid mechanics problems behave in a nonlinear manner and as a result superposition principles do not apply in fluid mechanics problems.

(x-momentum)

$$\rho\left[\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right]=-\frac{\partial p}{\partial x}+\rho g_x+\mu\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right] \tag{2.3}$$

(y-momentum)

$$\rho\left[\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right]=-\frac{\partial p}{\partial y}+\rho g_y+\mu\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right] \tag{2.4}$$

(z-momentum)

$$\rho\left[\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right]=-\frac{\partial p}{\partial z}+\rho g_z+\mu\left[\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right] \tag{2.5}$$

where

- ρ = density of the fluid.
- μ = dynamic viscosity of the fluid.
- p = pressure in the fluid. p is a function of space and time, $p\,(x, y, z, t)$.
- g_x, g_y, g_z = component of gravity in the x, y and z directions.

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2.3 Heat Transfer Equations

Convective heat transfer equations contain a portion of conduction in them. As a result the conductive heat transfer equations will be presented first followed by the convective heat transfer equation.

2.3.1 Conductive Heat transfer

The conductive heat transfer equation result from the application of Fourier's law of heat conduction combined with the conservation of heat flux to give,

$$\rho c_p \frac{\partial T}{\partial t} = k\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right] + \dot{Q} \tag{2.6}$$

Where

- c_p = specific heat of the solid substance.
- ρ = density of the solid substance.
- k = conductivity of the solid substance.
- $\dot{Q}$ = internal heat generation term.

The conduction equation is a linear equation as long as c_p, ρ, and k do not vary with temperature. In this case numerous analytical solutions exist for specific geometries. We shall solve this equation using the finite element formulation for instructional purposes.

2.3.2 Convective Heat Transfer

As it turns out, the only difference between convective and conductive heat transfer is fluid particles move through the media and take heat with them. This effect is modeled through the convective terms. Heat is also transferred in fluids by conduction as well. In fact, for flow of liquid metals (high conductivity) the conduction heat transfer can often be greater than the convective heat transfer. We shall investigate this effect later in this text using the finite element method.

For incompressible flows the energy equation may be written as,

$$\rho c_p \frac{\partial T}{\partial t} + \rho c_p \underbrace{\left[u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}\right]}_{\text{Convective terms}} = k \underbrace{\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right]}_{\text{Conduction terms}} + \mu\Phi + \dot{Q} \tag{2.7}$$

Where Φ is the viscous dissipation term,

$$\begin{aligned}\Phi = \Bigg[\underbrace{\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2}_{} + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2 \\ + 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right]\Bigg]\end{aligned} \tag{2.8}$$

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The viscous dissipation act like a heat generation term in equation 2.7. In effect the viscosity will extract energy from the flow filed and direct this energy towards a temperature increase in the fluid. Careful observation of the viscous dissipation term reveals an interesting fact, the energy is extracted only if there exist a *shear stain rate* in the fluid.

2.3.3 Buoyancy Driven Flows

Hot air rises while cold air sinks. This is the fact we all have observed either in the cold winter or scorching hot summer months. This phenomenon occurs for other fluids as well as air and is very predictable. Buoyancy driven flows have a new source term in the momentum equations which will account for the density change as a function of the temperature. In this case the momentum equations may be adjusted with this source term as follows,

(x-momentum)

$$\rho_o\left[\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right]=-\frac{\partial\left(p-p_o\right)}{\partial x}+\gamma_x\beta\left(T-T_o\right)+\mu\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right] \tag{2.9}$$

(y-momentum)

$$\rho_o\left[\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right]=-\frac{\partial\left(p-p_o\right)}{\partial y}+\gamma_y\beta\left(T-T_o\right)+\mu\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right] \tag{2.10}$$

(z-momentum)

$$\rho_o\left[\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right]=-\frac{\partial\left(p-p_o\right)}{\partial z}+\gamma_z\beta\left(T-T_o\right)+\mu\left[\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right] \tag{2.11}$$

Where

- Specific weight in the x-direction $\Rightarrow \gamma_x=\rho_o g_x$
- Specific weight in the y-direction $\Rightarrow \gamma_y=\rho_o g_y$
- Specific weight in the z-direction $\Rightarrow \gamma_z=\rho_o g_z$
- The coefficient of thermal expansion $\Rightarrow \beta=-\left(1/\rho\right)\left(\partial\rho/\partial T\right)_p$
- p_o, ρ_o and T_o are the pressure, density and temperature at a reference point.

Although the major change occurs in the momentum equations, there is some change to the energy equation as well,

$$\rho_o c_p\frac{\partial T}{\partial t}+\rho_o c_p\left[u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}\right]=k\left[\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right]+\mu\Phi+\dot{Q} \tag{2.12}$$

In most simulation it is assumed that the thermal expansion is a constant. However, this may not be the case. In fact, for an ideal gas, $\rho=p/RT$ and

$$\beta=-\frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_p=\frac{1}{\rho}\frac{p}{RT^2}=\frac{1}{T} \tag{2.13}$$

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As we can see, even for an ideal gas the thermal expansion coefficient is a function of the temperature. If the temperature difference through the flow domain is small, the assumption of a constant thermal expansion coefficient may be justified. We should keep this fact in mind as we make assumptions before our simulations.

2.4 Gradient Theorem

Given a scalar function $F(x, y, z)$, the gradient theorem may be stated as,

$$\boxed{\int_z \int_y \int_x \vec{\nabla} F dx\ dy\ dz = \oint_S F \vec{n} dS} \tag{2.14}$$

Where

$$\vec{\nabla} F = \left(\frac{\partial F}{\partial x} \vec{e}_x + \frac{\partial F}{\partial y} \vec{e}_y + \frac{\partial F}{\partial z} \vec{e}_z \right) \tag{2.15}$$

$$\begin{aligned} \int_z \int_y \int_x \left(\frac{\partial F}{\partial x} \vec{e}_x + \frac{\partial F}{\partial y} \vec{e}_y + \frac{\partial F}{\partial z} \vec{e}_z \right) dx\ dy\ dz &= \oint_S F \vec{n} dS \\ &= \oint_S \left(n_x \vec{e}_x + n_y \vec{e}_y + n_z \vec{e}_z \right) F dS \end{aligned} \tag{2.16}$$

Where the terms n_x, n_y and n_z are components of the normal vector around the surface of the volume.

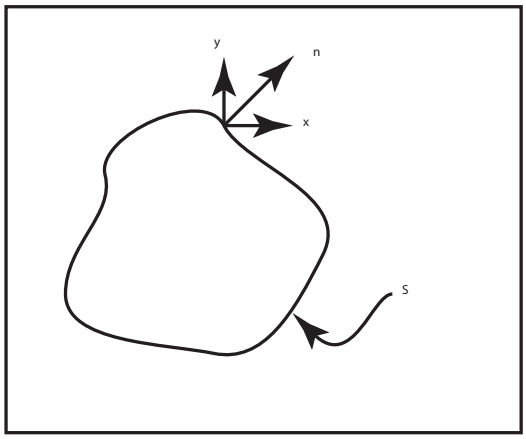


Figure 2.1: Normal Vector

Hence this equation may be broken into coordinate parts by observing the coefficients of

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the bases vectors, $\vec{e}_x$, $\vec{e}_y$ and $\vec{e}_z$.

$$\begin{aligned}\int_z\int_y\int_x\left(\frac{\partial F}{\partial x}\right)dx\,dy\,dz &= \oint_S n_x F dS\\ \int_z\int_y\int_x\left(\frac{\partial F}{\partial y}\right)dx\,dy\,dz &= \oint_S n_y F dS\\ \int_z\int_y\int_x\left(\frac{\partial F}{\partial z}\right)dx\,dy\,dz &= \oint_S n_z F dS\end{aligned} \tag{2.17}$$

2.5 Divergence Theorem

Given a vector function $\vec{G}\,(x,y,z)$, the divergence theorem of this vector field may be written as,

$$\boxed{\int_z\int_y\int_x \vec{\nabla}\cdot\vec{G}\,dx\,dy\,dz = \oint_S \vec{n}\cdot\vec{G}dS} \tag{2.18}$$

Where the divergence of the vector $\vec{G}\,(x,y,z)$ may be written as,

$$\begin{aligned}\vec{\nabla}\cdot\vec{G} &= \left(\frac{\partial}{\partial x}\vec{e}_x+\frac{\partial}{\partial y}\vec{e}_y+\frac{\partial}{\partial z}\vec{e}_z\right)\cdot\left(G_x\vec{e}_x+G_y\vec{e}_y+G_z\vec{e}_z\right)\\ &= \left(\frac{\partial G_x}{\partial x}+\frac{\partial G_y}{\partial y}+\frac{\partial G_z}{\partial z}\right)\end{aligned} \tag{2.19}$$

integrated over the domain, The divergence theorem may be stated as,

$$\int_z\int_y\int_x\left(\frac{\partial G_x}{\partial x}+\frac{\partial G_y}{\partial y}+\frac{\partial G_z}{\partial z}\right)dx\,dy\,dz = \oint_S\left(n_xG_x+n_yG_y+n_zG_z\right)dS \tag{2.20}$$

2.6 Towards Green's Identities

If we consider a function F may consist of a product of two functions $f\,(x,y,z)$ and $g\,(x,y,z)$ then the gradient theorem may be written as,

$$F = g\times f \tag{2.21}$$

Then

$$\int_z\int_y\int_x\vec{\nabla}\,(gf)\,dx\,dy\,dz = \oint_S(gf)\,\vec{n}dS \tag{2.22}$$

However,

$$\vec{\nabla}\,(gf) = f\vec{\nabla}g+g\vec{\nabla}f \tag{2.23}$$

Hence the gradient theorem may be written as,

$$\boxed{\int_z\int_y\int_x f\vec{\nabla}g dx\,dy\,dz = -\int_z\int_y\int_x g\vec{\nabla}f dx\,dy\,dz+\oint_S(gf)\,\vec{n}\,dS} \tag{2.24}$$

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We expand this equation to,

$$\begin{aligned}\int_z\int_y\int_x f\left(\frac{\partial g}{\partial x}\vec{e}_x+\frac{\partial g}{\partial y}\vec{e}_y+\frac{\partial g}{\partial z}\vec{e}_z\right)dx\,dy\,dz=\\ -\int_z\int_y\int_x g\left(\frac{\partial f}{\partial x}\vec{e}_x+\frac{\partial f}{\partial y}\vec{e}_y+\frac{\partial f}{\partial z}\vec{e}_z\right)dx\,dy\,dz\\ +\oint_S(gf)\left(n_x\vec{e}_x+n_y\vec{e}_y+n_z\vec{e}_z\right)\,dS\end{aligned} \tag{2.25}$$

By extracting each component of this equation we have

$$\begin{aligned}\int_z\int_y\int_x f\frac{\partial g}{\partial x}dx\,dy\,dz&=-\int_z\int_y\int_x g\frac{\partial f}{\partial x}dx\,dy\,dz+\oint_S(gf)\,n_x\,dS\\ \int_z\int_y\int_x f\frac{\partial g}{\partial y}dx\,dy\,dz&=-\int_z\int_y\int_x g\frac{\partial f}{\partial y}dx\,dy\,dz+\oint_S(gf)\,n_y\,dS\\ \int_z\int_y\int_x f\frac{\partial g}{\partial z}dx\,dy\,dz&=-\int_z\int_y\int_x g\frac{\partial f}{\partial z}dx\,dy\,dz+\oint_S(gf)\,n_z\,dS\end{aligned} \tag{2.26}$$

We shall use these equation extensively in the finite element method later.

2.7 Green's First Identity

The divergence theorem may be used if we consider the vector field $\vec{G}$ in equations 2.19 and 2.20 to be the product of a scalar function times the gradient of another scalar function,

$$\vec{G}(x,y,z)=g(x,y,z)\,\vec{\nabla}f(x,y,z) \tag{2.27}$$

With the divergence theorem written as,

$$\int_z\int_y\int_x\left[\nabla\cdot\vec{G}\right]dx\,dy\,dz=\oint_S\vec{n}\cdot\vec{G}\,dS \tag{2.28}$$

The divergence of G may be written as,

$$\left[\vec{\nabla}\cdot\vec{G}\right]=\vec{\nabla}\cdot\left[g\vec{\nabla}f\right]=\vec{\nabla}g\cdot\vec{\nabla}f+g\nabla^2f \tag{2.29}$$

Inserting this relationship into our divergence equation gives,

$$\int_z\int_y\int_x\vec{\nabla}g\cdot\vec{\nabla}f\,dx\,dy\,dz+\int_z\int_y\int_x g\nabla^2f\,dx\,dy\,dz=\oint_S\vec{n}\cdot\left[g\vec{\nabla}f\right]\,dS \tag{2.30}$$

Rearranging,

$$\int_z\int_y\int_x g\nabla^2f\,dx\,dy\,dz=-\int_z\int_y\int_x\vec{\nabla}g\cdot\vec{\nabla}f\,dx\,dy\,dz+\oint_S g\vec{n}\cdot\vec{\nabla}f\,dS \tag{2.31}$$

Or

$$\boxed{\int_z\int_y\int_x g\nabla^2f\,dx\,dy\,dz=-\int_z\int_y\int_x\vec{\nabla}g\cdot\vec{\nabla}f\,dx\,dy\,dz+\oint_S g\frac{\partial f}{\partial n}\,dS} \tag{2.32}$$

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Where the boundary term may be written as,

$$g\,\vec{n}\cdot\vec{\nabla}f = g\frac{\partial f}{\partial n} \tag{2.33}$$

Where the last term indicates a derivative normal to the surface,

$$\frac{\partial f}{\partial n} = n_x\frac{\partial f}{\partial x} + n_y\frac{\partial f}{\partial y} + n_z\frac{\partial f}{\partial z} \tag{2.34}$$

Equations 2.24 and 2.31 will be used to develop the finite element representation of our fluid and heat transfer equations.

2.8 Green's Second Identity

Green's second identity is included here for completeness. It is often used in boundary element methods and can be used in finite element methods if we wish to construct a weighting function with special properties.

Since both function $f(x,y,z)$ and $g(x,y,z)$ may be interchanged to give a different vector function $\vec{G}$, we develop two forms of Green's first identity.

$$\int_z\int_y\int_x g\nabla^2 f\;dx\;dy\;dz = -\int_z\int_y\int_x \vec{\nabla}g\cdot\vec{\nabla}f\;dx\;dy\;dz + \oint_S g\vec{n}\cdot\vec{\nabla}f\;dS \tag{2.35}$$

and

$$\int_z\int_y\int_x f\nabla^2 g\;dx\;dy\;dz = -\int_z\int_y\int_x \vec{\nabla}f\cdot\vec{\nabla}g\;dx\;dy\;dz + \oint_S f\vec{n}\cdot\vec{\nabla}g\;dS \tag{2.36}$$

If we subtract equation 2.35 from 2.36 we obtain,

$$\int_z\int_y\int_x \left(g\nabla^2 f - f\nabla^2 g\right)\;dx\;dy\;dz = \oint_S \left(g\vec{n}\cdot\vec{\nabla}f - f\vec{n}\cdot\vec{\nabla}g\right)\;dS \tag{2.37}$$

Or written in terms of normal derivatives,

$$\boxed{\int_z\int_y\int_x \left(g\nabla^2 f - f\nabla^2 g\right)\;dx\;dy\;dz = \oint_S \left(g\frac{\partial f}{\partial n} - f\frac{\partial g}{\partial n}\right)\;dS} \tag{2.38}$$

This is Green's second identity. Which could be used to develop special functions g, if f is our approximation to our differential equation.

2.9 Integration-By-Parts

Integration by parts is actually a one dimensional version of Green identities. In this case our functions depend on only one space dimension, $f(x)$ and $g(x)$. We begin by observing the product rule for these two functions.

$$d(gf) = gdf + gdf \tag{2.39}$$

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We may rearrange our equation to a more convenient form.

$$gdf = d(gf) - fdg \tag{2.40}$$

We may use the chain rule to reexpress our equation as,

$$f\frac{dg}{dx}dx = d(fg) - g\frac{df}{dx}dx \tag{2.41}$$

We now integrate this equation, as in the finite element method to obtain,

$$\int_{x_l}^{x_r} f\frac{dg}{dx}dx = -\int_{x_l}^{x_r} g\frac{df}{dx}dx + \int d(fg) \tag{2.42}$$

The last term on the right hand side may be thought of as a single variable $z = fg$. As a result, the integration $\int dz = z$ would yield.

$$\boxed{\int_{x_l}^{x_r} f\frac{dg}{dx}dx = -\int_{x_l}^{x_r} g\frac{df}{dx}dx + [fg]_{x_l}^{x_r}} \tag{2.43}$$

By comparing this equation with equations 2.24 and 2.26 we notice the similarity. First we must not the surface integral term in equation 2.24 and 2.26 correspond the last term on the right hand side of equation 2.43. This similarity is in fact an equality when we realize the surface of a one dimensional problem is actually the end points, x_l and x_r. We will used the integration by parts in the next chapter for the development of our one dimensional test problems.

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Chapter 3

1-D Finite Element Method

This chapter focuses on the development of the finite element method for one dimensional problems. We begin by examining one dimensional conduction heat transfer with a source term. This problem will be used to introduce several concepts in the finite element method. These concepts will be used and expanded on in multi-dimensional problems, in later chapters. Next, we begin a study into convective and conductive heat transfer problem in one dimensions. We shall expose a weakness in the standard finite element for convective-diffusive type problems. However, alterations to the standard finite element method remedy this problem. We next show how a time dependent formulation may be used to remedy this weakness as well, and in many cases this time dependent formulation is preferable.

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3.1 1-D Heat Transfer Model Problem

We begin with a model equation taken from heat transfer in one dimension. This heat transfer equation, with a source term may be written as,

$$k\frac{d^2T}{dx^2} + S = 0 \tag{3.1}$$

The differential equation itself is not sufficient without a set of boundary conditions. For this model case, we shall use homogeneous essential boundary conditions on both ends of our domain.

$$\begin{aligned} T\left(x=0\right) &= 0 \\ T\left(x=1\right) &= 0 \end{aligned} \tag{3.2}$$

Our source term S will be defined as,

$$S = -25\sin\left(4\pi x\right) - 33\sin\left(3\pi x\right) \tag{3.3}$$

Although this source term appears arbitrary it does allow a Fourier series solution for comparison and accuracy determination.

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3.1.1 Boundary Condition Types

Before progressing any further, it will be informative to categorize boundary condition types. Our model equation may be used to model many types of physical behavior. However, throughout this text, we will focus primarily on fluid mechanics and heat transfer problems. It should be apparent that our model problem is one dimensional heat transfer. In light of this fact there are two possible boundary conditions possible for this problem, temperature and heat flux. The heat flux is commonly expressed by using Fourier law,

$$q_x = -k\frac{dT}{dx} \tag{3.4}$$

In this case, as well as all cases throughout this text, the *essential boundary condition* is the primary variable, the temperature. The *natural boundary condition* is the heat flux. A simple case will show why the terminology of essential seems appropriate. Take a case of one dimensional heat transfer through a wall with no heat generation term, shown below.

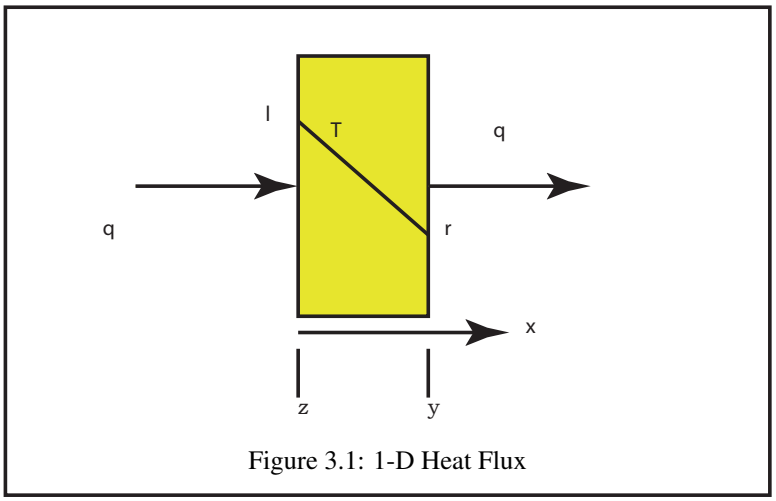


Figure 3.1: 1-D Heat Flux

Since the solution to this problem is a linear profile, our heat flux would be,

$$q = -k\frac{(T_r - T_l)}{(x_r - x_l)} \tag{3.5}$$

If we only specify the heat flux on both ends of the domain,

$$q|_{x=x_l} = q|_{x=x_r} \tag{3.6}$$

we would have a serious problem. As long as the difference in temperatures across the domain were the same, the heat flux would be the same. Therefore, any temperature combination would satisfy the heat flux as long as the difference between the two end temperatures is the same. As a result, it is *essential* that a *temperature* be specified at least on one side of the domain. Hence the term, *essential boundary condition*. As we shall see later in this chapter, the finite element method will reveal the essential and natural boundary conditions if we resort to an integration by parts to weaken functional requirement.

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3.1.2 Fourier Series, Exact, Solution

Usually, a junior or senior level advanced mathematics course will cover series solution to differential equations. As a result, we shall rely on our past contact with this solution type to develop an exact solution to our model problem. We begin by assuming a form of the solution, a Fourier series representation:

$$T(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + \sum_{n=1}^{\infty} b_n \sin(n\pi x) \tag{3.7}$$

We begin by taking two derivatives of this assumed solution:

$$\frac{d^2T}{dx^2} = -\sum_{n=1}^{\infty} n^2\pi^2 a_n \cos(n\pi x) - \sum_{n=1}^{\infty} n^2\pi^2 b_n \sin(n\pi x) \tag{3.8}$$

Inserting these terms into our differential equation, we have.

$$\begin{aligned} -k\left[\sum_{n=1}^{\infty} n^2\pi^2 a_n \cos(n\pi x) + \sum_{n=1}^{\infty} n^2\pi^2 b_n \sin(n\pi x)\right] \\ -25\sin(4\pi x) - 33\sin(3\pi x) = 0 \end{aligned} \tag{3.9}$$

Our goal here is to determine the constants of our Fourier series, a_n and b_n. By observation, the coefficients in front of our cosine terms, a_n, must all be zero. This is due to the absence of any other cosine terms in the equation. This leaves only the equation:

$$-k\sum_{n=1}^{\infty} n^2\pi^2 b_n \sin(n\pi x) - 25\sin(4\pi x) - 33\sin(3\pi x) = 0 \tag{3.10}$$

This equation may be satisfied for all x if the coefficients on the sine term in the following two equations is zero, all other b_n are zero.

$$\begin{aligned} \left(-k3^2\pi^2 b_3 - 33\right)\sin(3\pi x) = 0 \\ \left(-k4^2\pi^2 b_4 - 25\right)\sin(4\pi x) = 0 \end{aligned} \tag{3.11}$$

The only nonzero b_n terms are,

$$\begin{aligned} b_3 = \frac{33}{3^2\pi^2 k} \\ b_4 = \frac{25}{4^2\pi^2 k} \end{aligned} \tag{3.12}$$

As a result the exact solution to our model equation is,

$$T_e = -\frac{25}{16\pi^2 k}\sin(4\pi x) - \frac{11}{3\pi^2 k}\sin(3\pi x) + \frac{a_0}{2} \tag{3.13}$$

Notice the constant term associated with our coefficient a_0 still exist in the approximation. This is where our boundary conditions enter to save the day. The only way our two end

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conditions, equation 3.2, may be met is if our constant a_0 is zero. As a result our exact solution becomes,

$$\boxed{T_e = -\frac{25}{16\pi^2 k}\sin(4\pi x) - \frac{11}{3\pi^2 k}\sin(3\pi x)} \tag{3.14}$$

The section is included as a review. We selected a model equation that had an exact solution. As a result we shall use this exact solution to compare our finite element solution to.

3.2 Integration by Parts, Weak Form

We begin our finite element method with the concept that our final solution may be very close to the exact solution, but it is in fact an approximate solution. We represent this approximate solution as,

$$T_a^e(x) = \sum_{i=1}^{N} T_i^e \, \psi_j^e(x) \tag{3.15}$$

Where,

- $T_i^e \Rightarrow$ is the value of the temperature at a specific location in element e, more on this later.
- $\psi^e(x) \Rightarrow$ is the shape (or basis) functions used within element e. For our one dimensional problems we shall limit this shape function to a linear element.

We now insert this approximate solution into our differential equation to form a residual function.

$$\mathcal{R}(x) = \sum_{e=1}^{N_e}\left[k\frac{d^2T_a^e}{dx^2} + S\right] \tag{3.16}$$

Where,

- $e \Rightarrow$ is the element number.
- $N_e \Rightarrow$ is the number of element in the domain.
- $\mathcal{R}(x) \Rightarrow$ is the residual or error of our approximation as a function of x.

Of course, if our approximate solution became the exact solution the residual would equal zero everywhere in the domain and we would obtain equation 3.1, our original differential equation. However, we must now determine a method of obtaining the unknowns, T_i^e, in our domain. To do this we force our residual to be orthogonal to our shape functions.

$$\int_{x_l}^{x_r} \psi_j^e(x)\,\mathcal{R}(x)\,dx = \sum_{e=1}^{N_e}\int_{x_1^e}^{x_2^e} \psi_j^e\left[k\frac{d^2T_a^e}{dx^2} + S\right]dx = 0 \tag{3.17}$$

Where

- x_l and $x_r \Rightarrow$ are the left end and right end of the domain.
- x_1^e and $x_2^e \Rightarrow$ are the left end and right end of element e.

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- In order to keep our equations compact, our functional dependencies were dropped in this equation, $\psi_j^e(x) \to \psi_j^e$ and $T_a^e(x) \to T_a^e$.

If we take a moment to focus on the term,

$$\int_{x_1^e}^{x_2^e} \psi_j^e \left(k \frac{d^2 T_a^e}{dx^2} \right) dx \tag{3.18}$$

We observe a similarity to equation 2.43. If we observe the following substitutions, $\psi_j^e = f$ and $\frac{d^2 T_a^e}{dx^2} = g$ the similarities are apparent. This term becomes,

$$\int_{x_1^e}^{x_2^e} \psi_j^e \left(k \frac{d^2 T_a}{dx^2} \right) dx = \left[\psi_j^e \underbrace{k \frac{dT_a}{dx}}_{=-q_x} \right]_{x_1^e}^{x_2^e} - \int_{x_1^e}^{x_2^e} k \frac{d\psi_j^e}{dx} \frac{dT_a}{dx} dx \tag{3.19}$$

Using Fourier's law we may write this equation as,

$$\int_{x_1^e}^{x_2^e} \psi_j^e \left(k \frac{d^2 T_a}{dx^2} \right) dx = - \left[\psi_j^e q_x \right]_{x_1^e}^{x_2^e} - \int_{x_1^e}^{x_2^e} k \frac{d\psi_j^e}{dx} \frac{dT_a}{dx} dx \tag{3.20}$$

As a result of this integration by parts our full weighted residual statement becomes.

$$\boxed{\sum_{e=1}^{N_e} \left\{ \int_{x_1^e}^{x_2^e} \left[\psi_j^e S - k \frac{d\psi_j^e}{dx} \frac{dT_a}{dx} \right] dx - \left[\psi_j^e q_x \right]_{x_1^e}^{x_2^e} \right\} = 0} \tag{3.21}$$

This form of the equation is often called the *weak form*. This terminology originates from the requirements on the approximation functions being weakened as a result of using integration by parts on the second order derivative term. This weakened form also gave way to the appearance of a boundary term in the equations. As we shall see later, this boundary term is applied at both ends of our one dimensional elements.

3.3 1-D Finite Element Method

In this section we introduce the finite element method. The finite element method is based on the weighted residual method with functions that are defined over finite segments of the domain. A plot of these functions for linear approximations are shown in figure 3.2.

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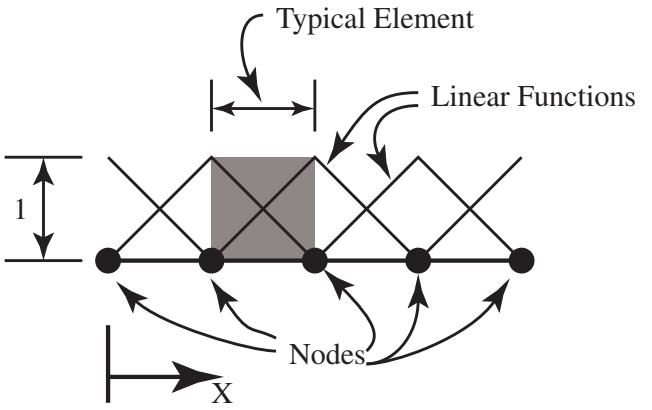


Figure 3.2: One Dimensional Finite Element Representation

Several items should be pointed out in this figure. First, the height of every function is unity. This added a significance to each of our unknowns in the series. For example, lets take an example where the temperature on these 5 nodes are, 0.5, 1.0, 2.0, 1.5, 0.5. This situation is shown in figure 3.3. Notice the linear function between each of these nodal values. In effect, we connect each nodal value with a straight line.

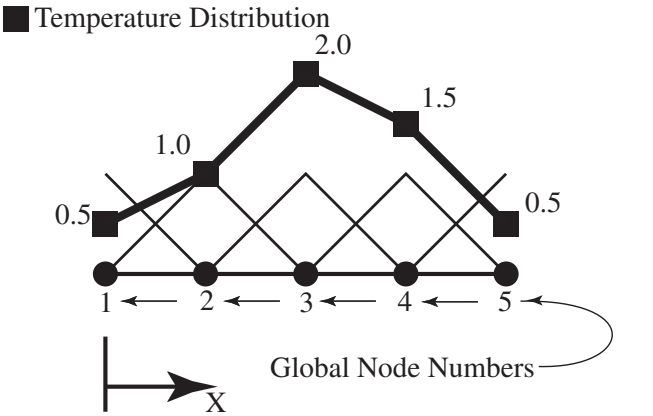


Figure 3.3: One Dimensional Temperature Distribution

From this figure, we can conclude that by multiplying each of our piecewise linear functions by the appropriate temperature we will obtain the temperature distribution. This is shown graphically in figure 3.4. If we add these 5 piecewise linear functions together we obtain our distribution shown in figure 3.3. To obtain these functions in figure 3.4 we simply multiplied our temperature times the unit valued functions shown in figure 3.2. The result of this multiplication of each piecewise function times the temperature and then summing the result gives the distribution shown in figure 3.3.

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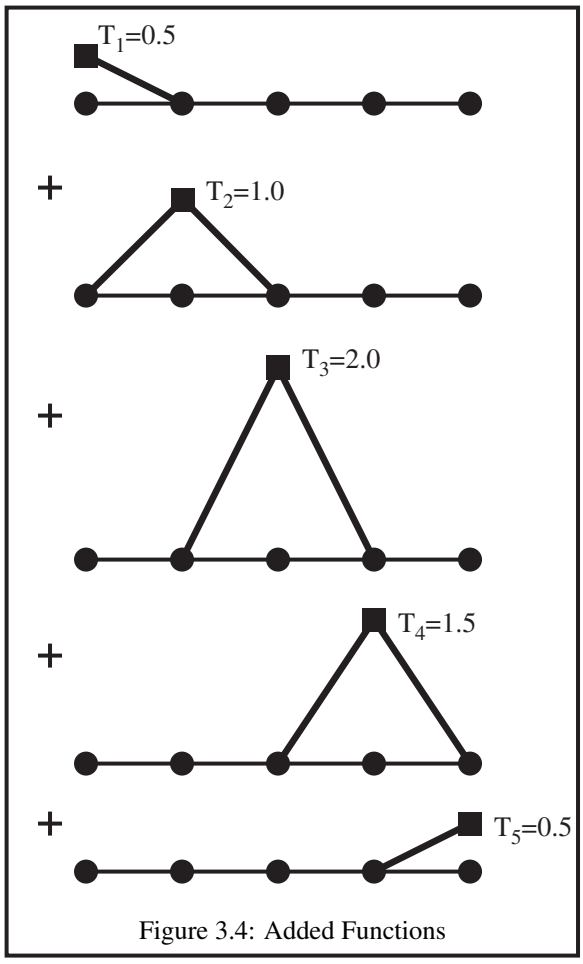


Figure 3.4: Added Functions

Since the entire domain is split into 4 elements, it is instructive to plot the domain with each element labeled as shown in figure 3.5.

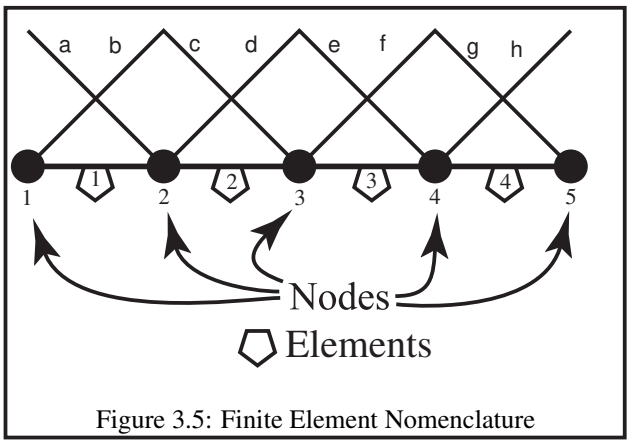


Figure 3.5: Finite Element Nomenclature

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In this figure we see the element numbers are sequential from left to right and placed inside the pentagons. The nodes are numbered globally from left to right as well. Each function is labeled, ψ_i^e, and is linear over each element. The functions, ψ_i^e, are written such that the subscript i indicated the local node within each element, and the superscript e indicated the element in which the basis function is in. By isolating one representative element, figure 3.6, we may investigate how to create a set of equations for each element.

Since each of these piecewise functions look the same regardless of which element we look at, we will pick out a representative element. If we look at a typical element we have an arrangement shown in figure 3.6. The left most node of this element is located at x_1^e and the rightmost node of this element is located at x_2^e. Notice a new coordinate system, ζ, originating from the middle of the element. This new coordinate will be called our local coordinate and is a value of $\zeta = -1$ at the left end of the element, local node 1. In addition, this local coordinate is a value of $\zeta = +1$ at local node number 2. This may seem arbitrary, however, later we will see there is some justification to this choice, especially when we develop higher order elements and multidimensional elements. The relationship between the physical coordinate and the local coordinate may be written as,

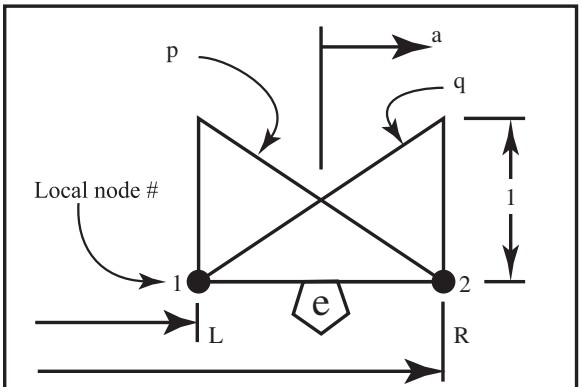


Figure 3.6: One Dimensional Finite Element

$$x(\zeta) = x_1^e \psi_1^e(\zeta) + x_2^e \psi_2^2(\zeta) \tag{3.22}$$

Where,

$$\psi_1^e(\zeta) = \frac{1}{2}(1-\zeta) \tag{3.23}$$

$$\psi_2^e(\zeta) = \frac{1}{2}(1+\zeta) \tag{3.24}$$

At first glance this may appear a bit cryptic. However, if we substitute equations 3.23 and 3.24 into equation 3.22 we obtain.

$$\begin{aligned} x(\zeta) =& \frac{x_1^e}{2}(1-\zeta) + \frac{x_2^e}{2}(1+\zeta) \\ =& \frac{x_1^e + x_2^e}{2} + \zeta\frac{x_2^e - x_1^e}{2} \\ =& x_m^e + \frac{l^e \zeta}{2} \end{aligned} \tag{3.25}$$

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Where l^e is the length of element e and x_m^e is the x value at the midpoint of the element. This equation satisfies our earlier constraints.

$$\begin{aligned} &\text{if} \quad \zeta = -1 \quad \text{then} \quad x = x_1^e \\ &\text{if} \quad \zeta = +1 \quad \text{then} \quad x = x_2^e \\ &\text{if} \quad \zeta = 0 \quad \text{then} \quad x = \frac{x_2^e + x_1^e}{2} \end{aligned} \tag{3.26}$$

Investigating the behavior of ψ_1^e and ψ_2^e is very instructive as well. Each function meets the following conditions.

$$\begin{aligned} \psi_1^e\left(\zeta = -1\right) &= 1 \\ \psi_1^e\left(\zeta = +1\right) &= 0 \end{aligned} \tag{3.27}$$

and

$$\begin{aligned} \psi_2^e\left(\zeta = -1\right) &= 0 \\ \psi_2^e\left(\zeta = +1\right) &= 1 \end{aligned} \tag{3.28}$$

Comparing the plots for our functions ψ_1^e and ψ_2^e we see our functions represented in equations 3.27 and 3.28 exactly match those in the figure 3.6.

3.3.1 Finite Elements of Model Problem

Returning to our heat transfer equation 3.1, 3.2 and 3.3, we now seek a finite element solution to this problem. We will work with linear elements, equation 3.23 and 3.24. The weighted residual statement was developed previously, equation 3.21 and is shown below.

$$\sum_{e=1}^{N_e} \left\{ \int_{x_1^e}^{x_2^e} \left[\psi_j^e S - k \frac{d\psi_j^e}{dx} \frac{dT_a}{dx} \right] dx - \left[\psi_j^e q_x \right]_{x_1^e}^{x_2^e} \right\} = 0 \tag{3.29}$$

We will focus our attention on the term in the brackets. This term is at the element level, figure 3.6.

$$\int_{x_1^e}^{x_2^e} \left[\psi_j^e S - k \frac{d\psi_j^e}{dx} \frac{dT_a}{dx} \right] dx = \left[\psi_j^e q_x \right]_{x_1^e}^{x_2^e} \tag{3.30}$$

$$\int_{x_1}^{x_2} \psi_i^e\left(x\right) \mathcal{R}\left(x\right) dx = 0 \qquad \text{for} \quad i = 1, 2 \tag{3.31}$$

Throughout this text the finite element methods we develop will all be of a weak form. As a result, we will weaken our weighted residual statement here as well. For our model problem this has already been accomplished, equation **??**. The only difference is the notation of our approximation function. Adjusting this equation to our new weighting functions, replacing f_i with ψ_i^e, we have,

$$\int_{x_1}^{x_2} \left[\psi_i^e S - \kappa \frac{d\psi_i^e}{dx} \frac{d\widetilde{T}}{dx} \right] dx - \left[\psi_i^e Q^e \right]_{x_1}^{x_2} = 0 \tag{3.32}$$

Also notice that our integration limits have been changed from x_l to x_1 and x_r to x_2. This changed is due to moving this integration to a element level where x_1 indicates the left most

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node in the element and x_2 indicated the right most node in the element. Recognizing that our dependent variable, in this case temperature, may be written as,

$$\widetilde{T}(x) = \sum_{j=1}^{N} \psi_j^e(x) T_j^e \tag{3.33}$$

Where T_i^e refers to the temperature at local node i within element e. Although we have written our basis function as $\psi_j^e(x)$, we shall from this reduce our notation to ψ_j^e understanding its functional dependency on x. We may insert this summation into our equation 3.31 to obtain the following equation.

$$\int_{x_1}^{x_2} \left[\kappa \frac{d\psi_i^e}{dx} \frac{d}{dx} \left(\sum_{j=1}^{N} \psi_j^e T_j^e \right) \right] dx = \int_{x_1}^{x_2} \psi_i^e S dx - [\psi_i^e Q^e]_{x_1}^{x_2} \tag{3.34}$$

Notice we rearranged our equation before inserting our approximation. If we recognize that summation and integration operation are interchangeable our equation becomes.

$$\sum_{j=1}^{N} \underbrace{\left[\int_{x_1}^{x_2} \kappa \frac{d\psi_i^e}{dx} \frac{d\psi_j^e}{dx} dx \right]}_{\textit{Element Matrix}} T_j^e = \int_{x_1}^{x_2} \psi_i^e S dx - [\psi_i^e Q^e]_{x_1}^{x_2} \tag{3.35}$$

The term labeled as *Element Matrix* may be written in matrix form as,

$$K_{ij}^e = \left[\int_{x_1}^{x_2} \kappa \frac{d\psi_i^e}{dx} \frac{d\psi_j^e}{dx} dx \right] \tag{3.36}$$

If we also represent the right hand side as a vector, equation 3.37, our equation may be written as shown in equation 3.38.

$$F_i^e = \int_{x_1}^{x_2} \psi_i^e S dx - [\psi_i^e Q^e]_{x_1}^{x_2} \tag{3.37}$$

$$\sum_{j=1}^{N} K_{ij}^e T_j^e = F_i^e \tag{3.38}$$

Often the summation notation shown in this equation is not written and simply implied. This is the case in tensor notation and we may use the rule of *summation on like indices*. If we use this notation our equation becomes,

$$K_{ij}^e T_j^e = F_i^e \tag{3.39}$$

We shall now investigate how to expand our matrices out. However, before we carry out this operation we must realize that we now have two independent coordinate, x and ζ. Since we need only one, we will investigate a coordinate transformation method in the next subsection.

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Coordinate Transformation

Our basis functions $\psi_1^e(\zeta)$ are written as functions of ζ, equations 3.23 and 3.24. However, the integrations shown in equation 3.34 are all integrate with respect to the x coordinate. We must transform one of these two coordinate systems into the other to carry out our integrations. We see that one coordinate system may be represented as a function of the other as shown in equation 3.25. We will now seek to transform our variable from x to ζ. We must transform our integration from one independent, x, coordinate to the other, ζ. In doing do we use the chain rule to obtain a differential express as shown below.

$$dx = \frac{dx}{d\zeta} d\zeta \tag{3.40}$$

We must also transform the limits of our integration. This may be done by observing the relationships shown in equation 3.26. As a result, the following integration relationships are equivalent.

$$\int_{x_1}^{x_2} [\cdots]\, dx = \int_{-1}^{+1} [\cdots]\, \frac{dx}{d\zeta} d\zeta \tag{3.41}$$

Since our coordinates are related by equation 3.25, equation 3.40 is easily determined. Taking the derivative of equation 3.25 we obtain,

$$\frac{dx}{d\zeta} = \frac{1}{2}\left(x_2^e - x_1^e\right) = \frac{l^e}{2} \tag{3.42}$$

Where l^e is the length of element e. Equation 3.41 may be rewritten as,

$$\int_{x_1}^{x_2} \underbrace{[\cdots]}_{Kernel} dx = \frac{l^e}{2} \int_{-1}^{+1} \underbrace{[\cdots]}_{Kernel} d\zeta \tag{3.43}$$

The term shown above in labeled as *Kernel* is now our focus. If this term originated from our K_{ij}^e matrix we must consider a transformation from the x coordinate to the ζ coordinate for the differential terms. This is achieved again by the chain rule, shown below.

$$\frac{d\psi_i^e}{dx}\frac{d\psi_j^e}{dx} = \left\{\frac{d\psi_i^e}{d\zeta}\frac{d\zeta}{dx}\right\}\left\{\frac{d\psi_j^e}{d\zeta}\frac{d\zeta}{dx}\right\} = \left(\frac{2}{l^e}\right)^2 \frac{d\psi_i^e}{d\zeta}\frac{d\psi_j^e}{d\zeta} \tag{3.44}$$

Where equation 3.42 was used to convert the $d\zeta/dx$ term. Combining equations 3.43 and 3.36 we have a stiffness matrix K_{ij}^e as,

$$K_{ij}^e = \frac{2}{l^e}\int_{-1}^{+1} \kappa \frac{d\psi_i^e}{d\zeta}\frac{d\psi_j^e}{d\zeta} d\zeta \tag{3.45}$$

Likewise, combining equation 3.43 with equation 3.37 we have the following equation.

$$F_i^e = \frac{l^e}{2}\int_{-1}^{+1} \psi_i^e S d\zeta + \left[\psi_i^e Q^e\right]_{-1}^{+1} \tag{3.46}$$

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Elemental Evaluations

In this section we shall determine the element stiffness equation, K_{ij}^e, and the element right hand side vector F_i^e. Referring to equations 3.23 and 3.24 we may easily determine the term $d\psi_j^e/d\zeta$.

$$\begin{aligned} \frac{d\psi_1^e}{d\zeta} &= -\frac{1}{2} \\ \frac{d\psi_2^e}{d\zeta} &= +\frac{1}{2} \end{aligned} \tag{3.47}$$

As a result it is very easy to determine the stiffness matrix K_{ij}^e from the combination of equations 3.47 and 3.45.

$$\begin{aligned} K_{11}^e &= \frac{2}{l^e}\int_{-1}^{+1} \kappa \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right) d\zeta = +\frac{\kappa^e}{2l^e}\int_{-1}^{+1} d\zeta = +\frac{\kappa^e}{l^e} \\ K_{12}^e &= \frac{2}{l^e}\int_{-1}^{+1} \kappa \left(-\frac{1}{2}\right)\left(+\frac{1}{2}\right) d\zeta = -\frac{\kappa^e}{2l^e}\int_{-1}^{+1} d\zeta = -\frac{\kappa^e}{l^e} \\ K_{21}^e &= \frac{2}{l^e}\int_{-1}^{+1} \kappa \left(+\frac{1}{2}\right)\left(-\frac{1}{2}\right) d\zeta = -\frac{\kappa^e}{2l^e}\int_{-1}^{+1} d\zeta = -\frac{\kappa^e}{l^e} \\ K_{22}^e &= \frac{2}{l^e}\int_{-1}^{+1} \kappa \left(+\frac{1}{2}\right)\left(+\frac{1}{2}\right) d\zeta = +\frac{\kappa^e}{2l^e}\int_{-1}^{+1} d\zeta = +\frac{\kappa^e}{l^e} \end{aligned} \tag{3.48}$$

We have made one assumption in this case, the conductivity, κ, may vary from element to element. Hence the conductivity carries a superscript identifying which element it relates to, κ^e. As a result of equation 3.48, our element level stiffness matrix may be written as,

$$[K] = \frac{\kappa^e}{l^e}\begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} \tag{3.49}$$

Our right hand side term, equation 3.46 contains two terms. As a result we shall investigate these terms separately we may rewrite equation 3.46 as,

$$F_i^e = \hat{F}_i^e - \check{F}_i^e \tag{3.50}$$

Where,

$$\hat{F}_i^e = \frac{l^e}{2}\int_{-1}^{+1} \psi_i^e S d\zeta \tag{3.51}$$

$$\check{F}_i^e = \left[\psi_i^e Q^e\right]_{-1}^{+1} = \psi_i^e\left(+1\right)\left[Q^e\right]_{+1} - \psi_i^e\left(-1\right)\left[Q^e\right]_{-1} \tag{3.52}$$

We shall first begin by investigating the general term $\check{F}_i^e$. This term exist regardless of the source term S. As a result this section is universal. If we notice the conditions listed in equations 3.27 and 3.28 we find this boundary term $\check{F}_i^e$ becomes,

$$\check{F}_1^e = \underbrace{\psi_1^e\left(+1\right)}_{=0}\left[Q^e\right]_{+1} - \underbrace{\psi_1^e\left(-1\right)}_{=1}\left[Q^e\right]_{-1} = -\left[Q^e\right]_{-1} = -Q_l^e \tag{3.53}$$

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$$\check{F}_2^e = \underbrace{\psi_2^e(+1)}_{=1}[Q^e]_{+1} - \underbrace{\psi_2^e(-1)}_{=0}[Q^e]_{-1} = +[Q^e]_{+1} = +Q_r^e \tag{3.54}$$

Where Q_l^e is the heat transfer at the left end of element e, and Q_r^e is the heat transfer at the right end of element e. At first glance these end conditions seem difficult to deal with. However, as we assemble the elements together these terms will cancel each other out element by element. It is useful at element matrix at this point.

$$\frac{\kappa^e}{l^e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1^e \\ T_2^e \end{Bmatrix} = \begin{Bmatrix} Q_l^e \\ -Q_r^e \end{Bmatrix} + \begin{Bmatrix} \hat{F}_1^e \\ \hat{F}_2^e \end{Bmatrix} \tag{3.55}$$

We shall wait until we are focused on our model problem to deal with the $\hat{F}_i^e$ term. This term is completely dependent on the source term and how it varies with the independent coordinate.

3.3.2 Elemental Assembly

In this section we deal with assembling the element matrices to form the overall global matrix. For instructive purposes we will begin with a two element situation. Multi element assembles will become very apparent from the two element assemblies. Figure 3.7 shows two elements in both separate form and assembled form.

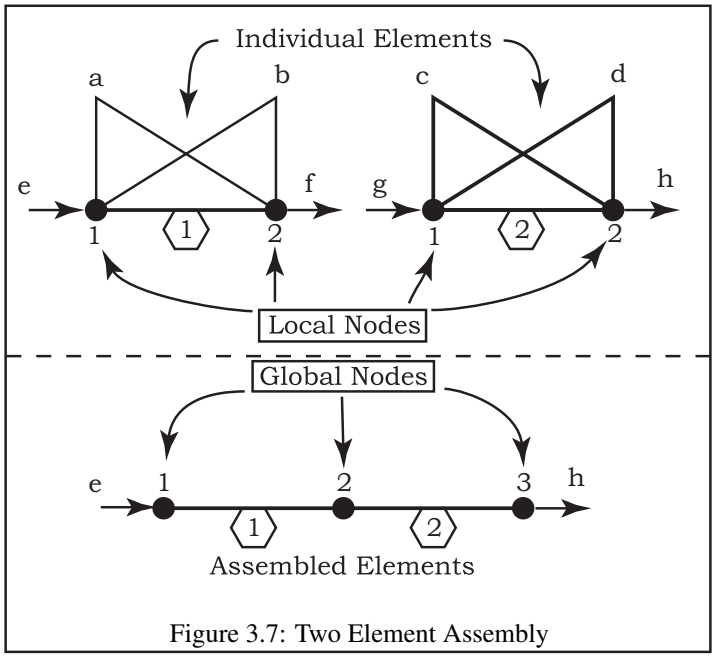


Figure 3.7: Two Element Assembly

By comparing the unassembled to the assembled elements we notice an interesting fact. The local nodes and the global nodes do not correspond directly. In fact we may develop a simple way of linking the local nodes of each element to the global node number. We shall use the same method used for two and three dimensional element assemblies for the present case. This

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method is commonly call a boolean or connectivity matrix. For our simple two element case this matrix may be seen in figure 3.8. In this connectivity matrix the global nodes are listed. The rows of this matrix are the elements while the columns represent the local nodes of the elements. Using this indexing method we may immediately determine the location of any local node in any element by looking at the entries in the corresponding row (element) and column (local node). Generally speaking for one dimensional elements we number the nodes from the left to the right. This will give matrices like those shown in the equation below.

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \\ \vdots & \vdots \\ N & N+1 \end{bmatrix} \tag{3.56}$$

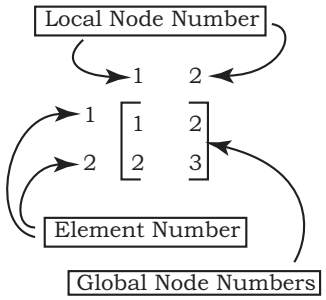


Figure 3.8: Connectivity Matrix

Where N is the number of elements in the element collection, grid. We shall see later how this connectivity matrix may be used to combine the element matrices together to form one global matrix.

For a two element assembly we have two individual element matrices, shown below.

$$\frac{\kappa^1}{l^1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1^1 \\ T_2^1 \end{Bmatrix} = \begin{Bmatrix} Q_l^1 \\ -Q_r^1 \end{Bmatrix} + \begin{Bmatrix} \hat{F}_1^1 \\ \hat{F}_2^1 \end{Bmatrix} \tag{3.57}$$

$$\frac{\kappa^2}{l^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1^2 \\ T_2^2 \end{Bmatrix} = \begin{Bmatrix} Q_l^2 \\ -Q_r^2 \end{Bmatrix} + \begin{Bmatrix} \hat{F}_1^2 \\ \hat{F}_2^2 \end{Bmatrix} \tag{3.58}$$

Using the connectivity matrix the local variables T_i^e may be transformed to global variables T_l. Using this simplification our equations may be written as shown in equation 3.59. Notice there has been an added row and column to each element matrix. This added row and column is in recognition of the existence of three unknown temperatures.

$$\frac{\kappa^1}{l^1}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} Q_l^1 \\ -Q_r^1 \\ 0 \end{Bmatrix} + \begin{Bmatrix} \hat{F}_1^1 \\ \hat{F}_2^1 \\ 0 \end{Bmatrix} \tag{3.59}$$

$$\frac{\kappa^2}{l^2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ Q_l^2 \\ -Q_r^2 \end{Bmatrix} + \begin{Bmatrix} 0 \\ \hat{F}_1^2 \\ \hat{F}_2^2 \end{Bmatrix} \tag{3.60}$$

Adding these two matrices together we obtain a single global matrix shown below.

$$\begin{bmatrix} \frac{\kappa^1}{l^1} & -\frac{\kappa^1}{l^1} & 0 \\ -\frac{\kappa^1}{l^1} & \frac{\kappa^1}{l^1}+\frac{\kappa^2}{l^2} & -\frac{\kappa^2}{l^2} \\ 0 & -\frac{\kappa^2}{l^2} & \frac{\kappa^2}{l^2} \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} Q_l^1 \\ Q_l^2 - Q_r^1 \\ -Q_r^2 \end{Bmatrix} + \begin{Bmatrix} \hat{F}_1^1 \\ \hat{F}_2^1 + \hat{F}_1^2 \\ \hat{F}_2^2 \end{Bmatrix} \tag{3.61}$$

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Of particular interest in this equation is the boundary term $Q_l^2 - Q_r^1$. This term involves the heat flux from the right side of element 1, Q_r^1, and the left side of element 2, Q_l^2. The heat flux from the right side of element 1 is the same as the heat flux from the left side of element 2. As a result the term $Q_l^2 - Q_r^1 = 0$. There is one exception to this, if there is an applied heat flux at this point, node. If the applied heat flux is Q_A, the equation becomes $Q_l^2 - Q_r^1 = Q_2^A$. In addition, if the conductivities, κ^e, and the length of each element, l^e is constant for each element the global system of equations become,

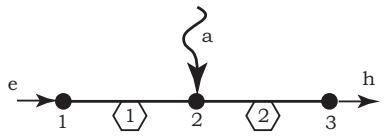


Figure 3.9: Point Heat Flux at Node 2

$$\frac{\kappa}{l}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} Q_l^1 \\ Q_2^A \\ -Q_r^2 \end{Bmatrix} + \begin{Bmatrix} \hat{F}_1^1 \\ \hat{F}_2^1 + \hat{F}_1^2 \\ \hat{F}_2^2 \end{Bmatrix} \tag{3.62}$$

From this assembly of two elements we may form a few general rules which will help us assemble element matrices in one, two and three dimensions. If we wrote our equations 3.57 and equation 3.58 in symbolic form our equations would become.

$$\begin{array}{c} \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} K_{11}^1 & K_{12}^1 \\ K_{21}^1 & K_{22}^1 \end{bmatrix} \end{array}\begin{Bmatrix} T_1^1 \\ T_2^1 \end{Bmatrix} = \begin{Bmatrix} F_1^1 \\ F_2^1 \end{Bmatrix} \tag{3.63}$$

$$\begin{array}{c} \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix} K_{11}^2 & K_{12}^2 \\ K_{21}^2 & K_{22}^2 \end{bmatrix} \end{array}\begin{Bmatrix} T_1^2 \\ T_2^2 \end{Bmatrix} = \begin{Bmatrix} F_1^2 \\ F_2^2 \end{Bmatrix} \tag{3.64}$$

Where the numbers above and to the left of the element matrix come directly from the connectivity matrix, figure 3.8. From element 1 the row of the connectivity matrix reveals the global node numbers of 1 and 2. For element 2, the second row of the connectivity reveals the global node numbers of 2 and 3. These rows in the connectivity matrix give direction as to where the element stiffness matrix will end up in the global stiffness matrix.

For one dimensional problems the global stiffness matrix becomes very predictable and looks exactly like that produced by a finite difference method. For a six element - seven node grid with constant element length and conductivity we obtain a global matrix shown below.

$$\frac{\kappa}{l}\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \end{Bmatrix} = \begin{Bmatrix} Q_l^1 \\ Q_2^A \\ Q_3^A \\ Q_4^A \\ Q_5^A \\ Q_6^A \\ -Q_r^6 \end{Bmatrix} + \begin{Bmatrix} \hat{F}_1^1 \\ \hat{F}_2^1 + \hat{F}_1^2 \\ \hat{F}_3^1 + \hat{F}_2^2 \\ \hat{F}_4^1 + \hat{F}_3^2 \\ \hat{F}_5^1 + \hat{F}_4^2 \\ \hat{F}_6^1 + \hat{F}_5^2 \\ \hat{F}_6^2 \end{Bmatrix} \tag{3.65}$$

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In this case there are point heat sources at all nodes, shown in figure 3.10. The point sources for the two end nodes are shown as Q_l^1 and Q_r^6 and must stay in the equations even if no nodal heat fluxes exist.

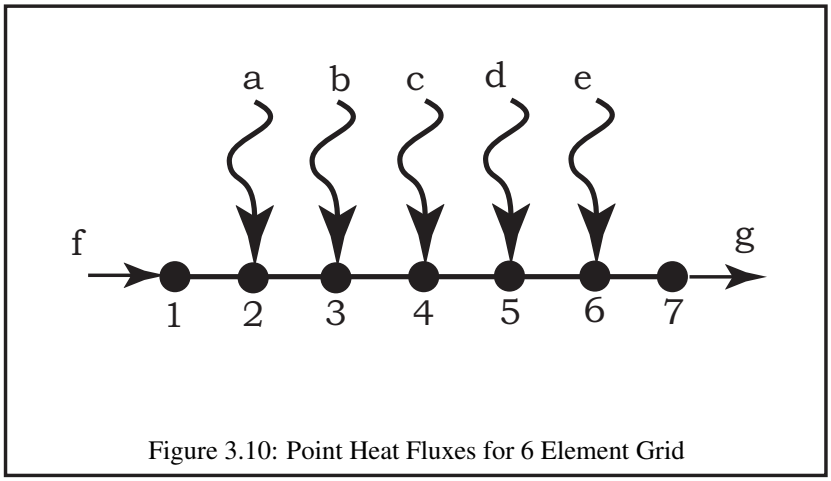


Figure 3.10: Point Heat Fluxes for 6 Element Grid

Through out this text, for the most part these point heat fluxes or loads will be zero. One case that may have these point heat fluxes is one dimensional modeling of a fin. In this case the heat flux will be a function of an ambient temperature and the current nodal temperature. However, the nodal heat fluxes at the left and rightmost nodes are not zero. These nodal heat fluxes are a result of an implied boundary condition. We will now investigate several boundary condition types for an example problem.

First, we look at the rank of our system matrix. The rank can be thought of as the number of independent equations that exist within a given set of equations. For example, equation 3.65 contains seven equations. If all seven equations are independent of each other the rank would be seven. The rank of this matrix is six and therefore, the matrix shown in equation 3.65 is singular. In fact the rank of any global stiffness matrix will be one less than its size. This means there are actually six independent equations existing within the seven equations shown. Therefore, we must come up with another equation to add to the six equations to obtain a complete set. To solve this problem we resort to our essential boundary condition, the specification of temperature either at the left or right of the domain, T_1 or T_7. Lets take the case where we have a temperature imposed on the left end of our domain. The first equation is simply removed and replace with a boundary condition, say $T_1 = 4.4$.

$$\frac{\kappa}{l}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \end{Bmatrix} = \begin{Bmatrix} 4.4 \\ Q_2^A \\ Q_3^A \\ Q_4^A \\ Q_5^A \\ Q_6^A \\ -Q_r^6 \end{Bmatrix} + \begin{Bmatrix} 0 \\ \hat{F}_2^1 + \hat{F}_1^2 \\ \hat{F}_3^1 + \hat{F}_2^2 \\ \hat{F}_4^1 + \hat{F}_3^2 \\ \hat{F}_5^1 + \hat{F}_4^2 \\ \hat{F}_6^1 + \hat{F}_5^2 \\ \hat{F}_6^2 \end{Bmatrix} \tag{3.66}$$

With this change the rank of our matrix becomes seven, therefore the matrix is no longer singular and may be inverted to obtain a solution for our temperatures. If our nodal heat

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source terms are all zero and our domain source term, equation 3.51, is zero, the system equation may be written as shown in equation 3.67.

$$\frac{\kappa}{l}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \end{Bmatrix} = \begin{Bmatrix} 4.4 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -Q_r^6 \end{Bmatrix} \tag{3.67}$$

In this case the only unknown boundary term is Q_r^6. This term is the heat flux from the right side of element six or the right end of the last element. Since our basic equation, equation 2.1, is second order, two boundary conditions are necessary for a well posed problem. We may specify the temperature on the left end of the domain and either the temperature or heat flux on the right end of the domain. If for example we specify the heat flux of 10.4, our system becomes

$$\frac{\kappa}{l}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \end{Bmatrix} = \begin{Bmatrix} 4.4 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 10.4 \end{Bmatrix} \tag{3.68}$$

Which is a well posed problem. In fact the only ill posed problem would be the specification of the heat flux on both end. In this case the matrix, equation 3.66, is singular and the system is not solvable. This is the reason specification of the temperature is often call *essential* boundary conditions while the heat flux specification is called the natural boundary condition.

Finite Element Solution of Test Case

In order to test the finite element method on our test case we must deal with our source term, equation 2.3. This source term is a function of the x coordinate while the integration we must perform, equation 3.51 is written in terms of the local element coordinate ζ. Combining these equations we obtain,

$$\hat{F}_i^e = \frac{l^e}{2}\int_{-1}^{+1} \psi_i^e \left[-25\sin\left(4\pi x\right) - 33\sin\left(3\pi x\right)\right] d\zeta \tag{3.69}$$

Making use of our equations relating these two coordinates, equation 3.25,

$$\hat{F}_i^e = \frac{l^e}{2}\int_{-1}^{+1} \psi_i^e \left[-25\sin\left(4\pi\left(x_m^e + \frac{l^e\zeta}{2}\right)\right) - 33\sin\left(3\pi\left(x_m^e + \frac{l^e\zeta}{2}\right)\right)\right] d\zeta \tag{3.70}$$

We shall use this relationship in our Matlab code to obtain the source loads in each of our elements.

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Finite Element Solution of Test Problem

Using Matlab we obtain several solutions to our problem. These solutions are shown in figure 3.11.

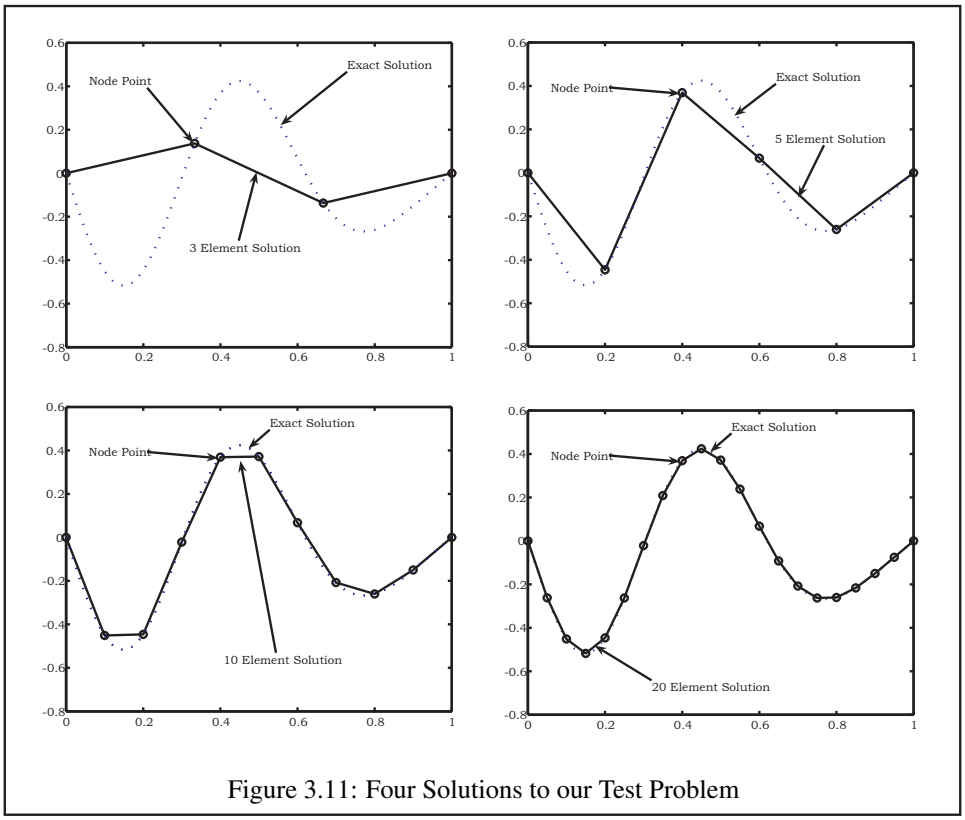


Figure 3.11: Four Solutions to our Test Problem

These plots are for a 3, 5, 10 and 20 element solutions. One observation that may be made is the node values related to the exact solution. In all cases the nodal point values lay on the exact solution. This is in fact always the case for one dimensional linear second order differential equation. As we can see our 20 element solution compares very well with our exact solution.

3.4 One Dimensional Convection-Diffusion

In this section we focus on a problem of particular interest to computational fluid dynamic, convective diffusion equation. We will focus on the one dimensional model problem first. Later, this problem will be expanded into two dimensions. The problem originates from convection and diffusion occurring in a one dimensional stream of fluid. One location such a one dimensional convection diffusion event occurs in a Jominy end quench test, figure 3.12. In these type of test a Jominy bar is removed from an oven at very high temperature and placed through a hole in a plate. Just below the end of the test section a water supply is turned on such

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that the water hits the end of the test section. The water is supplied at a constant temperature of around 60 to 80 degrees F. The intense heat from the Jominy bar can diffuse back up the stream, which the cool water is convected along the x direction. Of course, this problem is unsteady due to the cooling of the bar by the stream of cool water. However, this case is only meant to show a type case of one dimensional convection diffusion. We will model this problem such that the temperature of the Jominy bar is constant. For this case the governing equation is shown below.

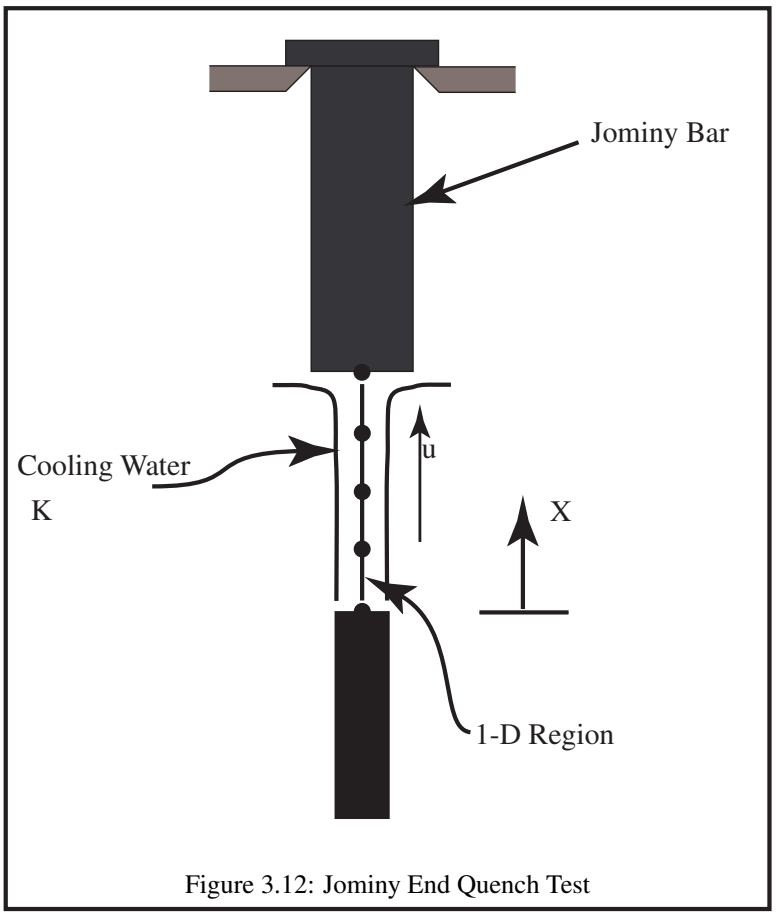


Figure 3.12: Jominy End Quench Test

$$u\frac{dT}{dx} - \kappa\frac{d^2T}{dx^2} = 0 \tag{3.71}$$

Subject to

$$\begin{aligned} T(0) &= 0 \\ T(1) &= 1 \end{aligned} \tag{3.72}$$

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Where, u is the velocity of the fluid, k is the thermal conductivity, and θ is the non dimensional temperature. The exact solution to this problem is,

$$T(x) = \frac{e^{Px} - 1}{e^{P} - 1} \tag{3.73}$$

where,

$$P = \frac{u}{\kappa} \tag{3.74}$$

As plot for differing values of P is shown in figure 3.13.

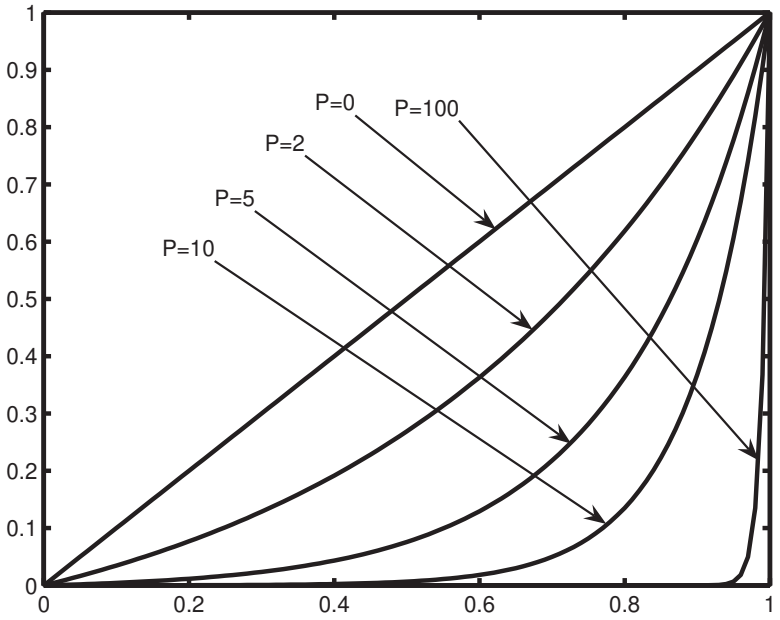


Figure 3.13: Exact Solutions for various values of P

In our case we seek a finite element solution to our equation 3.71. If we cast this equation into a weighted residual form our element equation is,

$$\int_{x_1}^{x_2} \psi_i^e \left(u\frac{d\widetilde{T}}{dx} - \kappa\frac{d^2\widetilde{T}}{dx^2} \right) dx \tag{3.75}$$

Integrating by parts our element equations become,

$$\int_{x_1}^{x_2} \left(\psi_i^e u\frac{d\widetilde{T}}{dx} + \kappa\frac{d\psi_i^e}{dx}\frac{d\widetilde{T}}{dx} \right) dx + [\psi_i^e Q^e]_{x_1}^{x_2} \tag{3.76}$$

Where

$$Q^e = -\kappa\frac{dT}{dx} \tag{3.77}$$

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Inserting the approximation for the temperature,

$$\widetilde{T}(x) = \sum_{j=1}^{N} \psi_j^e(x) T_j^e \tag{3.78}$$

we have,

$$\int_{x_1}^{x_2} \left(\psi_i^e u \frac{d}{dx} \left(\sum_{j=1}^{N} \psi_j^e T_j^e \right) + \kappa \frac{d\psi_i^e}{dx} \frac{d}{dx} \left(\sum_{j=1}^{N} \psi_j^e T_j^e \right) \right) dx + [\psi_i^e Q^e]_{x_1}^{x_2} \tag{3.79}$$

Or combining terms we have,

$$\sum_{j=1}^{N} \left[\int_{x_1}^{x_2} \left(\psi_i^e u \frac{d\psi_j^e}{dx} + \kappa \frac{d\psi_i^e}{dx} \frac{d\psi_j^e}{dx} \right) dx \right] T_j^e + [\psi_i^e Q^e]_{x_1}^{x_2} \tag{3.80}$$

It is obvious that our element stiffness equation originates from the term shown below.

$$K_{ij} = \int_{x_1}^{x_2} \left(\psi_i^e u \frac{d\psi_j^e}{dx} + \kappa \frac{d\psi_i^e}{dx} \frac{d\psi_j^e}{dx} \right) dx \tag{3.81}$$

Where κ and u are constants. Transforming into local coordinates our element stiffness equation becomes,

$$K_{ij} = \int_{-1}^{+1} \left(\psi_i^e u \frac{d\psi_j^e}{d\zeta} \frac{d\zeta}{dx} + \kappa \frac{d\psi_i^e}{d\zeta} \frac{d\psi_j^e}{d\zeta} \frac{d\zeta}{dx} \frac{d\zeta}{dx} \right) \frac{dx}{d\zeta} d\zeta \tag{3.82}$$

Simplifying,

$$K_{ij} = u \int_{-1}^{+1} \psi_i^e \frac{d\psi_j^e}{d\zeta} d\zeta + \kappa \int_{-1}^{+1} \frac{d\psi_i^e}{d\zeta} \frac{d\psi_j^e}{d\zeta} \frac{d\zeta}{dx} d\zeta \tag{3.83}$$

Or,

$$K_{ij} = u \int_{-1}^{+1} \psi_i^e \frac{d\psi_j^e}{d\zeta} d\zeta + \underbrace{\frac{2\kappa}{l^e} \int_{-1}^{+1} \frac{d\psi_i^e}{d\zeta} \frac{d\psi_j^e}{d\zeta} d\zeta}_{\text{Equation 3.45}} \tag{3.84}$$

Since the second term on the right hand side has already been determined we shall focus on the other term.

$$\widehat{K}_{ij} = u \int_{-1}^{+1} \psi_i^e \frac{d\psi_j^e}{d\zeta} d\zeta \tag{3.85}$$

Inserting the values for our basis functions, equations 3.23, 3.24, and 3.47 we may break our matrix into its components,

$$\begin{aligned}
\widehat{K}_{11}^e &= u \int_{-1}^{+1} \frac{1}{2}(1-\zeta) \left(-\frac{1}{2} \right) d\zeta = -\frac{u}{4} \int_{-1}^{+1} (1-\zeta) d\zeta = -\frac{u}{2} \\
\widehat{K}_{12}^e &= u \int_{-1}^{+1} \frac{1}{2}(1-\zeta) \left(+\frac{1}{2} \right) d\zeta = +\frac{u}{4} \int_{-1}^{+1} (1-\zeta) d\zeta = +\frac{u}{2} \\
\widehat{K}_{21}^e &= u \int_{-1}^{+1} \frac{1}{2}(1+\zeta) \left(-\frac{1}{2} \right) d\zeta = -\frac{u}{4} \int_{-1}^{+1} (1+\zeta) d\zeta = -\frac{u}{2} \\
\widehat{K}_{22}^e &= u \int_{-1}^{+1} \frac{1}{2}(1+\zeta) \left(+\frac{1}{2} \right) d\zeta = +\frac{u}{4} \int_{-1}^{+1} (1+\zeta) d\zeta = +\frac{u}{2}
\end{aligned} \tag{3.86}$$

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Therefore our element stiffness matrices are,

$$[K] = \underbrace{\frac{u^e}{2}\begin{bmatrix} -1 & +1 \\ -1 & +1 \end{bmatrix}}_{\text{Convection}} + \underbrace{\frac{\kappa^e}{l^e}\begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix}}_{\text{Diffusion}} \tag{3.87}$$

Looking at a typical 5 element assembly, with no boundary conditions applied we have.

$$\frac{u^e}{2}\begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} + \frac{\kappa^e}{l^e}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} Q_l^1 \\ 0 \\ -Q_r^2 \end{Bmatrix} \tag{3.88}$$

Stripping out an equation for any internal node we have,

$$\frac{u^e}{2}(T_3 - T_1) - \frac{\kappa^e}{l^e}(T_1 - 2T_2 + T_3) = 0 \tag{3.89}$$

Note, in this case there is only one internal node. However, the equation for all internal nodes is the same regardless of the number of elements in the grid. The general form of the equation becomes,

$$\frac{u^e}{2}(T_{k+1} - T_{k-1}) - \frac{\kappa^e}{l^e}(T_{k-1} - 2T_k + T_{k+1}) = 0 \tag{3.90}$$

The general solution to these sets of equations become,

$$T_k = A_0 + B_0 \underbrace{\left[\frac{1+\beta}{1-\beta}\right]^k}_{\textit{Term A}} \tag{3.91}$$

Where A_0 and B_0 are constants used to satisfy the given boundary conditions and β is given below.

$$\beta = \frac{ul^e}{2\kappa^e} \tag{3.92}$$

An interesting situation arises when $\beta = 1$. In this case the denominator of *Term A* becomes zero, and as a result the term itself becomes infinite. However, the solution itself is possible. An example of this solution is shown in figure 3.14.. In fact this case will always result in an interesting behavior. The solution will be zero everywhere except at the right most node, where the boundary condition forces the temperature to unity. Of course this solution is not the correct solution and is a special case just before solution instability occurs. If we allow β to increase above a value of 1 we have another interesting result. In this case oscillations begin to occur in the solution itself. An example of this is shown in the figure

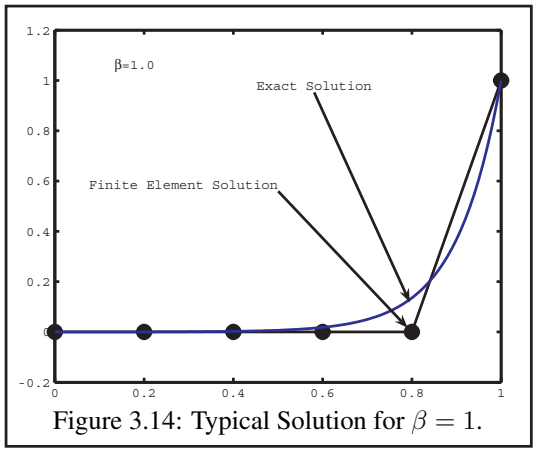


Figure 3.14: Typical Solution for $\beta = 1$.

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3.15. This oscillation occurs due to
the behavior of the *Term A* in equation 3.91. The behavior of this term may be written in an equation form as shown below

$$\text{If } \beta > 1 \text{ then } \left[\frac{1+\beta}{1-\beta}\right] < 0 \text{ and } \left[\frac{1+\beta}{1-\beta}\right]^k \Longrightarrow \begin{cases} > 0 & k \text{ even} \\ < 0 & k \text{ odd} \end{cases} \tag{3.93}$$

Since the sign of *Term A* alternates this is a clear indication of an instability. Likewise, when $\beta < 1$ the sign of *Term A* does not change and as a result the solution is stable, see equation 3.94 below.

$$\text{If } \beta < 1 \text{ then } \left[\frac{1+\beta}{1-\beta}\right] > 0 \text{ and } \left[\frac{1+\beta}{1-\beta}\right]^k > 0 \quad \forall \ k \tag{3.94}$$

The plot in figure 3.15 shows a situation with $\beta > 1$. Notice the oscillation in the solution. By comparing our solutions to those for our second order test problem we observe a major difference, our test problem obtained nodal point solutions which lay directly on exact solution, figure 3.11. Not only does this not occur in the convection diffusion problem but in certain cases the solution is unstable. This leads us to search out a method which lays directly on the exact solution.

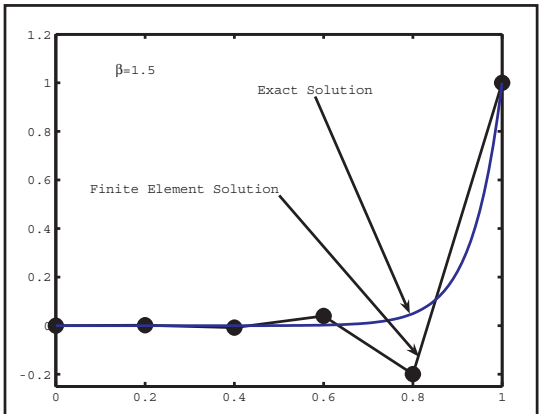


Figure 3.15: Typical Solution for $\beta = 1.5$.

Matlab Code

The Matlab code for the one dimensional convection diffusion cases are shown in figures 3.16 to 3.20.

```
 1 %% Finite Element of 1-D Convective Diffucsion (Copyright Dr R. Fithen)
 2 %  Filename: FEM1DConvDiff.m
 3 %
 4     clear
 5 % Number of elements
 6     Nelem=5;
 7 % Length of each element
 8     le=1.0/Nelem;
 9 %conductivity
10     kappa=1.0;
11 % Use beta to determine the value of u
12     beta=1.0;
13     u=beta*(2/le);
14     P=u/kappa;
```

Figure 3.16: Convection Diffusion Section #1

In section #1 of the listing we simply determine the constants for our problem. Line 6 specifies the number of elements and line 8 determines the length

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of each element based on the number of elements and the domain size of 1.0. Line 10 and 12 specify the constants κ and β. The velocity u is determined on line 13, equation 3.92. Once these constants are determined, line 16 specifies the diffusion portion of the element stiffness matrix, equation 3.87. Line 18 specifies the portion of the element stiffness which originates from the convection term. While line 19 simply combines these two matrices into one element stiffness matrix. Lines 21 and 22 create an element connectivity matrix, equation 3.56 and figure 3.8. This connectivity matrix is used to direct the assembly on our elements into their proper locations in the global stiffness matrix. This assemble occurs in lines 25, 26 and 27. These lines of code will place the element stiffness matrix directly into the global stiffness matrix exactly like what occurs in equations 3.59 and 3.60. To watch this process in Matlab, one has only to set a break point at line 26 and take the semicolon off the end of this line. Stepping through the code from this point we can watch the construction of the global stiffness matrix.

```
15 %Element Matrix
16      Ke=kappa/le*[1 -1;-1 1];
17 % Convective portion
18      Kec=u/2*[-1 1;-1 1];
19      Ke=Ke+Kec;
20 % connectivity matrix...
21      conn(:,1)=1:Nelem;
22      conn(:,2)=2:Nelem+1;
```

Figure 3.17: Convection Diffusion Section #2

```
23 %Build global matrix
24      Kg=zeros(Nelem+1,Nelem+1);
25      for i=1:Nelem
26          Kg(conn(i,:),conn(i,:))=Kg(conn(i,:),conn(i,:))+Ke;
27      end
```

Used to direct element contributions to the global stiffness matrix

Figure 3.18: Convection Diffusion Section #3

In section #4 of the listing our boundary conditions are specified. Line 29 clears our row 1 of our global stiffness matrix, while line 30 places a 1 in the diagonal. Line places a 0 on the right hand side of the system matrix. This is exactly what was described in altering the global stiffness matrix, equation 3.65, from its original form to a form shown in equation 3.66. As a result of these alterations the first equation of our global system matrix is removed and replaced by our boundary equation, $T_1 = 0$. Lines 32 through 34 do the same except, the boundary equation is $T_{\text{Nelem}+1} = 1$. This specifies the temperature of 1 on the right most node in the grid.

```
28 % Place Boundary Conditions
29      Kg(1,:)=0;
30      Kg(1,1)=1;
31      Fg(1,1)=0;
32      Kg(Nelem+1,:)=0;
33      Kg(Nelem+1,Nelem+1)=1;
34      Fg(Nelem+1,1)=1;
```

Figure 3.19: Convection Diffusion Section #4

Once these matrix equations have been assembled and boundary conditions implied,

```
35 % Solve for temperature
36      T=inv(Kg)*Fg;
37 % Plot our Solution
38      x=0:le:1;
39      x=x';
40      plot(x,T,'-ko')
41      hold on
42 % Exact Solution (Plot)
43      xp=0:.01:1;
44      Te=(exp(P*xp)-1)/(exp(P)-1);
45      figure(1)
46      plot(xp,Te,':')
47      hold on
```

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the solution is obtained in line 36. Once the solution is obtained lines 38 to 41 plot this finite element solution to our problem. Line 38 specifies the location of each of our nodes, while line 39 simply transposes the vector into the proper form for our plot command, line 40. Line 41 forces our plot window to hold the current plot and allow other plotting to be placed in this window without deleting the present plot. Lines 43 to 46 plot the exact solution on top of our finite element solution. Line 43 generates a sufficient number of independent coordinate locations to allow for a smooth representation of the exact solution. Line 44 generates the exact solution while line 46 plots the solution. The code is available from the book's web site and readers are encouraged to download and run these codes.

3.4.1 Petrov-Galerkin Formulation

With a sufficient number of elements Rayleigh Ritz based finite elements will obtain a good solution for the convection diffusion problem. However, as the convection contribution increases the solution becomes unstable. This alone is motivation to go in search of an alternate formulation that will be stable regardless of the element size and convective contribution level. Although this method is not the final method we advocate, it is instructive to observe how this method is constructed. This method is similar and a slight derivative of that of Brooks and Hughes, [**?**]. We begin by adapting our original equation, 3.75, to a Petrov-Galerkin statement.

$$\int_{x_1}^{x_2} \varpi_i^e \left(u\frac{d\widetilde{T}}{dx} - \kappa\frac{d^2\widetilde{T}}{dx^2} \right) dx \tag{3.95}$$

Or

$$\int_{x_1}^{x_2} \left(\varpi_i^e u\frac{d\widetilde{T}}{dx} + \kappa\frac{d\varpi_i^e}{dx}\frac{d\widetilde{T}}{dx} \right) dx - [\varpi_i^e Q^e]_{x_1}^{x_2} \tag{3.96}$$

Where the function ϖ_i^e is not the basis function used in the approximation, ψ_i^e. In fact this term may be written as,

$$\varpi_i^e = \psi_i^e + \frac{\alpha l^e}{2}\frac{d\psi_i^e}{dx} \tag{3.97}$$

Where α is a yet to be determined constant. The derivative of this weighting function is,

$$\frac{d\varpi_i^e}{dx} = \frac{d\psi_i^e}{dx} + \frac{\alpha l^e}{2}\underbrace{\frac{d^2\psi_i^e}{dx^2}}_{=0} \tag{3.98}$$

Since the approximation terms ψ_i^e are linear their second derivative is zero while their first derivative is a constant. As a result equation 3.98 becomes,

$$\frac{d\varpi_i^e}{dx} = \frac{d\psi_i^e}{dx} \tag{3.99}$$

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Inserting equations 3.97 and 3.99 into equation 3.96 we obtain,

$$\int_{x_1}^{x_2}\left(\left(\psi_i^e+\frac{\alpha l^e}{2}\frac{d\psi_i^e}{dx}\right)u\frac{d\widetilde{T}}{dx}+\kappa\frac{d\psi_i^e}{dx}\frac{d\widetilde{T}}{dx}\right)dx-\left[\left(\psi_i^e+\frac{\alpha l^e}{2}\frac{d\psi_i^e}{dx}\right)Q^e\right]_{x_1}^{x_2} \tag{3.100}$$

Rearranging this equation we obtain,

$$\int_{x_1}^{x_2}\left(\psi_i^e u\frac{d\widetilde{T}}{dx}+\left(\kappa+\frac{u\alpha l^e}{2}\right)\frac{d\psi_i^e}{dx}\frac{d\widetilde{T}}{dx}\right)dx-\left[\left(\psi_i^e+\frac{\alpha l^e}{2}\frac{d\psi_i^e}{dx}\right)Q^e\right]_{x_1}^{x_2} \tag{3.101}$$

Inserting the approximation for temperature, equation 3.78, into our equation we obtain,

$$\sum_{j=1}^{N}\left[\int_{x_1}^{x_2}\left(\underbrace{\psi_i^e u\frac{d\psi_j^e}{dx}}_{C_{ij}^e}+\underbrace{\left(\kappa+\frac{u\alpha l^e}{2}\right)\frac{d\psi_i^e}{dx}\frac{d\psi_j^e}{dx}}_{D_{ij}^e}\right)dx\right]T_j^e-\left[\left(\psi_i^e+\frac{\alpha l^e}{2}\frac{d\psi_i^e}{dx}\right)Q^e\right]_{x_1}^{x_2} \tag{3.102}$$

Our element level equations become

$$C_{ij}^eT_j^e+D_{ij}^eT_j^e=F_i^e \tag{3.103}$$

Where C_{ij}^e is our convective contribution and D_{ij}^e is our diffusive contribution. Notice what has occurred here, a portion of our convective contribution has moved into the diffusive term. This contribution, $u\alpha l^e/2$, is a function of the velocity, u. As the convective contribution increases by increasing the velocity, u, so to does the diffusive contribution. One could consider this added diffusion as *false diffusion*, a term commonly used in the past. However, as we shall see in our simulations this so called *false diffusion* term may be used to obtain the exact solution at our nodes. We shall not use this term, *false diffusion*, to describe the added diffusion term due to its connotation that somehow this added term is not correct. Our Convective and diffusive matrices are shown below.

$$[C^e]=\frac{u}{2}\begin{bmatrix}-1&1\\-1&1\end{bmatrix} \tag{3.104}$$

$$[D^e]=\left(\frac{u\alpha}{2}+\frac{\kappa}{l^e}\right)\begin{bmatrix}1&-1\\-1&1\end{bmatrix} \tag{3.105}$$

Combining these element matrices into a two element configuration our global system equation becomes,

$$\frac{u}{2}\begin{bmatrix}-1&1&0\\-1&0&1\\0&-1&1\end{bmatrix}\begin{Bmatrix}T_{k-1}\\T_k\\T_{k+1}\end{Bmatrix}+\left(\frac{u\alpha}{2}+\frac{\kappa}{l^e}\right)\begin{bmatrix}1&-1&0\\-1&2&1\\0&-1&1\end{bmatrix}\begin{Bmatrix}T_{k-1}\\T_k\\T_{k+1}\end{Bmatrix}=\begin{Bmatrix}F_{k-1}\\F_k\\F_{k+1}\end{Bmatrix} \tag{3.106}$$

Pulling our our internal nodal equation assuming no source term at node 2 we obtain a general equation for all internal node as,

$$\frac{u}{2}\left(T_{k+1}-T_{k-1}\right)-\left(\frac{u\alpha}{2}+\frac{\kappa}{l^e}\right)\left(T_{k-1}-2T_k+T_{k+1}\right)=0 \tag{3.107}$$

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Or

$$\beta\left(T_{k+1}-T_{k-1}\right)-\left(\beta\alpha+1\right)\left(T_{k-1}-2T_k+T_{k+1}\right)=0 \tag{3.108}$$

Where

$$\beta=\frac{ul^e}{2\kappa} \tag{3.109}$$

The numerical solution to this equation becomes,

$$T_k=A_0+B_0\left[\frac{1+\beta\left(\alpha+1\right)}{1-\beta\left(\alpha-1\right)}\right]^k \tag{3.110}$$

If in this case we decide that $\alpha=1$ the solution is completely free of oscillations. In fact, $\alpha=1$ corresponds to a fully upwind finite difference scheme. Christie [**?**] has shown if we equate our exact solution to the nodal solution an optimal value for α may be established as,

$$\alpha=\coth\beta-\frac{1}{\beta} \tag{3.111}$$

An example solution is shown in figure 3.21

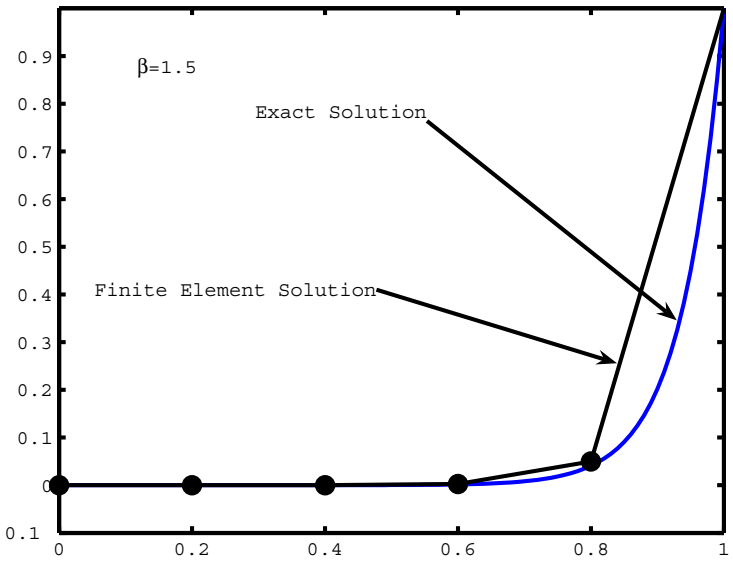


Figure 3.21: Petrov-Galerkin Finite Element Solution

Regardless of the velocity the nodal solution fall directly on the exact solution, much like our pure diffusion problem. However, we should not get a feeling of overconfidence at this point. As it turns out this exact match is not possible in two and three dimensional problems. Therefore, we should not place too much weight on this Petrov-Galerkin method. In fact the only purpose for its inclusion is to study the appearance of the diffusion matrix D^e_{ij} in equation 3.102. This altered matrix adds stability to an otherwise unstable solution. If we can generate another formulation that will produce a diffusion matrix like the one in equation 3.102, our solution will become stable in one, two and three dimensions. As we shall see later, this is very possible.

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Petrov-Galerkin Matlab Code

The Matlab code for our Petrov-Galerkin method is only slightly different from our Matlab code from the last section. In fact, there are only two lines of code that must be changed. These lines are 15 and 17 as shown in the figure 3.22. We simply add a line defining α, equation 3.111. We also add a contribution to the diffusion matrix on line 17, equation 3.105.

```
 1 %% Finite Element of 1-D Convective Diffucsion (Copyright Dr R. Fithen)
 2 %  Filename: FEM1DConvDiffPG.m
 3 % Uses Petrov Galerkin Method
 4     clear
 5 % Number of elements
 6     Nelem=5;
 7 % Length of each element
 8     le=1.0/Nelem;
 9 %conductivity
10     kappa=1.0;
11 % Use beta to determine the value of u
12     beta=1.5;
13     u=beta*(2/le);
14     P=u/kappa;
15     alpha=coth(beta)-1/beta;
16 %Element Matrix
17     Ke=(kappa/le+u*alpha/2)*[1 -1;-1 1];
```


Figure 3.22: Petrov-Galerkin Matlab Code

3.5 Fractional Step - Lax Wendroff Method

Although the Petrov-Galerkin method works very well for our one dimensional convection diffusion equation, there are some eventual problems that we shall encounter. First, our goal is to move into two and three dimensions and there are complications with respect to our weighting functions that will occur with this move. Secondly and more important, it is our desire to be able to predict time dependent flow situations in two and three dimensions. This leads us to a new area which deals with a marching a solution through time. As it turns out, how we decide to march our solution can actually stabilize our solution such that an altered weighting function is not required. We will begin by looking at our one dimensional case with a time derivative added back to our equation.

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} - \kappa\frac{\partial^2 T}{\partial x^2} = 0 \tag{3.112}$$

We will solve our equation by first splitting the equation into two separate equations each having its own characteristics. In one dimensions we will be able to recombine these equations to see how accurate our approach may be. However, in higher dimensions this recombination is very difficult. With this in mind we will separate our equations and later recombine and observe their accuracy. Our split equations are,

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} = 0 \tag{3.113}$$

and

$$\frac{\partial T}{\partial t} - \kappa\frac{\partial^2 T}{\partial x^2} = 0 \tag{3.114}$$

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These two equations are solved sequentially through time alternating between each equation as our time step increases. Equation 3.113 contains pure convection while equation 3.114 containers pure diffusion. We shall make use of each equations characteristics to develop a time stepping approach.

Convection Step

In this subsection we focus on the convection fractional step. Although there are many different approaches to modeling a pure convection step, we will choose a method call the Lax-Wendroff method. Our reasoning for this choice is its similarity to some aspects of our Petrov-Galerkin formulation as we shall see shortly. We begin with a Taylor series expansion of out temperature about a specific point in time as,

$$T\left(x,t+\Delta t\right)=T\left(x,t\right)+\Delta t\left[\frac{\partial T}{\partial t}\right]_{x,t}+\frac{\left(\Delta t\right)^{2}}{2!}\left[\frac{\partial^{2}T}{\partial t^{2}}\right]_{x,t}\underbrace{+\sum_{n=3}^{\infty}\frac{\left(\Delta t\right)^{n}}{n!}\left[\frac{\partial^{n}T}{\partial t^{n}}\right]_{x,t}}_{\text{Truncate}} \tag{3.115}$$

Although the series continues to infinity, we shall truncate our series and leave a second order term. Rearranging our convetive step equation we have,

$$\frac{\partial T}{\partial t}=-u\frac{\partial T}{\partial x} \tag{3.116}$$

Taking the time derivative of this equation we have,

$$\frac{\partial^{2}T}{\partial t^{2}}=-\frac{\partial}{\partial t}\left\{u\frac{\partial T}{\partial x}\right\} \tag{3.117}$$

In this case our velocity is constant and the differential may be reversed. In addition, we insert equation 3.116 in for the first order time differential to give,

$$\frac{\partial^{2}T}{\partial t^{2}}=-u\frac{\partial}{\partial x}\left\{\frac{\partial T}{\partial t}\right\}=u^{2}\frac{\partial^{2}T}{\partial x^{2}} \tag{3.118}$$

Inserting equations 3.116 and 3.118 into equation 3.115 we obtain a second order approximation of a Taylor series expansion as,

$$T\left(x,t+\Delta t\right)=T\left(x,t\right)-u\Delta t\left[\frac{\partial T}{\partial x}\right]_{x,t}+\frac{\left(\Delta t\right)^{2}u^{2}}{2}\left[\frac{\partial^{2}T}{\partial x^{2}}\right]_{x,t} \tag{3.119}$$

Notice the last term in this equation. This term looks very much like a diffusion term. The appearance of this term in the formulation stabilizes the solution. This diffusion term is very similar to what happened in our Petrov-Galerkin formulation earlier. At this point we case our equations into a Galerkin form. Multiplying our equation by a weighting function, ψ_i^e we obtain,

$$\int_{x_l}^{x_r}\psi_i^e T^{n+1/2}dx=\int_{x_l}^{x_r}\psi_i^e\left[T^{n}-u\Delta t\frac{\partial T^{n}}{\partial x}+\frac{\left(\Delta t\right)^{2}u^{2}}{2}\frac{\partial^{2}T^{n}}{\partial x^{2}}\right]dx \tag{3.120}$$

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Where,

$$\begin{aligned} T^{n+1/2} &= T\left(x, t+\Delta t\right) \\ T^{n} &= T\left(x, t\right) \end{aligned} \tag{3.121}$$

Integrating the last term in equation 3.120 by parts we obtain,

$$\begin{aligned} \int_{x_l}^{x_r} \psi_i^e T^{n+1/2} dx = & \int_{x_l}^{x_r} \psi_i^e T^n dx - \int_{x_l}^{x_r} \psi_i^e u \Delta t \frac{\partial T^n}{\partial x} dx \\ & - \int_{x_l}^{x_r} \frac{\left(\Delta t\right)^2 u^2}{2} \frac{\partial \psi_i^e}{\partial x} \frac{\partial T^n}{\partial x} dx \\ & + \left[\frac{\left(\Delta t\right)^2 u^2}{2} \psi_i^e \frac{\partial T^n}{\partial x} \right]_{x_l}^{x_r} \end{aligned} \tag{3.122}$$

If we substitute our finite element approximation,

$$T^n = \sum_{j=1}^{N} \psi_j^e T_j^{e(n)} \tag{3.123}$$

into our equation 3.120 and integrate by parts the se we obtain,

$$\begin{aligned} \sum_{j=1}^{N} \left[\int_{x_l}^{x_r} \psi_i^e \psi_j^e dx \right] T_j^{e(n+1/2)} = & \sum_{j=1}^{N} \left[\int_{x_l}^{x_r} \psi_i^e \psi_j^e dx \right] T_j^{e(n)} \\ & - \sum_{j=1}^{N} \left[\int_{x_l}^{x_r} \psi_i^e u \Delta t \frac{\partial \psi_j^e}{\partial x} dx \right] T_j^{e(n)} \\ & - \sum_{j=1}^{N} \left[\int_{x_l}^{x_r} \frac{\left(\Delta t\right)^2 u^2}{2} \frac{\partial \psi_i^e}{\partial x} \frac{\partial \psi_j^e}{\partial x} dx \right] T_j^{e(n)} \\ & + \left[\psi_i^e \mathcal{Q}^n \right]_{x_l}^{x_r} \end{aligned} \tag{3.124}$$

Where

$$\mathcal{Q}^n = \frac{\left(\Delta t\right)^2 u^2}{2} \frac{\partial T^n}{\partial x} \tag{3.125}$$

In an elemental matrix form this equation may be written as,

$$\left[\mathcal{M}^e\right] \left\{T^e\right\}^{n+1/2} = \left[\mathcal{M}^e\right] \left\{T^e\right\}^{n} - \left[\mathcal{C}^e\right] \left\{T^e\right\}^{n} - \left[\mathcal{D}^e\right] \left\{T^e\right\}^{n} + \left\{F^e\right\}^{n} \tag{3.126}$$

Where,

$$\mathcal{M}_{ij}^e = \int_{x_l}^{x_r} \psi_i^e \psi_j^e dx \tag{3.127}$$

$$\mathcal{C}_{ij}^e = u \Delta t \int_{x_l}^{x_r} \psi_i^e \frac{\partial \psi_j^e}{\partial x} dx \tag{3.128}$$

$$\mathcal{D}_{ij}^e = \frac{\left(\Delta t\right)^2 u^2}{2} \int_{x_l}^{x_r} \frac{\partial \psi_i^e}{\partial x} \frac{\partial \psi_j^e}{\partial x} dx \tag{3.129}$$

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Notice the time requirements in equations 3.119 and 3.126. The temperature at the new time step is on the left side of the equal sign while all information on the right is at the previous time step. Situations were the dependent variable exist by itself on the left hand side of the equation are call *explicit* methods. Situations where more than one term on exist on the left hand side of the equations are call *implicit* methods. As we shall see in the next section the numerical method for our diffusion equation is implicit.

Diffusion Step

The diffusion time step portion begins with our equation 3.114. If we begin with this equation and cast it into a weighted residual form out equation becomes,

$$\int_{x_l}^{x_r} \psi_i^e \left[\frac{\partial T}{\partial t} - \kappa \frac{\partial^2 T}{\partial x^2} \right] dx = 0 \tag{3.130}$$

Integrating our weighted residual statement by parts we obtain,

$$\int_{x_l}^{x_r} \psi_i^e \frac{\partial T}{\partial t} dx + \int_{x_l}^{x_r} \kappa \frac{\partial \psi_i^e}{\partial x} \frac{\partial T}{\partial x} dx = \left[\kappa \psi_i^e \frac{\partial T}{\partial x} \right]_{x_l}^{x_r} \tag{3.131}$$

If we substitute our approximation for the temperatures as,

$$\begin{aligned} T &= \sum_{j=1}^{N} \psi_j^e T_j^e \\ \frac{\partial T}{\partial t} &= \sum_{j=1}^{N} \psi_j^e \frac{\partial T_j^e}{\partial t} = \sum_{j=1}^{N} \psi_j^e \dot{T}_j^e \end{aligned} \tag{3.132}$$

In essence what we have done with this approximation is assume that our basis function, ψ_j^e, are only dependent on the space coordinate, $\psi_j^e(x)$. Where the temperatures at the nodes are functions of time, $T_j^e(t)$. This is exactly what we do for separable functions in mathematics to obtain eigenfunctions. Substituting our approximations, equation 3.132, into our weak form, equation 3.131, we obtain,

$$\sum_{j=1}^{N} \left[\int_{x_l}^{x_r} \psi_i^e \psi_j^e dx \right] \dot{T}_j^e + \sum_{j=1}^{N} \left[\int_{x_l}^{x_r} \kappa \frac{\partial \psi_i^e}{\partial x} \frac{\partial \psi_j^e}{\partial x} \right] T_j^e dx = - \left[\psi_i^e Q \right]_{x_l}^{x_r} \tag{3.133}$$

Where,

$$Q = -\kappa \frac{\partial T}{\partial x} \tag{3.134}$$

In matrix form this equation becomes,

$$[M^e] \left\{ \dot{T}^e \right\} + [K^e] \{T^e\} = \{F^e\} \tag{3.135}$$

The time difference method used on this equation is a standard Crank-Nicolson method. This method is second order accurate, like that of the Lax-Wendroff method, and may be expressed as,

$$\frac{\{T^e\}^{n+1} - \{T^e\}^{n+1/2}}{\Delta t} = \frac{1}{2} \left\{ \dot{T}^e \right\}^{n+1/2} + \frac{1}{2} \left\{ \dot{T}^e \right\}^{n+1} \tag{3.136}$$

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In order to use this equation we must express our matrix equation, 3.135, at two different points in time.

$$\begin{aligned}[M^e]\left\{\dot{T}^e\right\}^{n+1} + [K^e]\{T^e\}^{n+1} &= \{F^e\}^{n+1} \\ [M^e]\left\{\dot{T}^e\right\}^{n+1/2} + [K^e]\{T^e\}^{n+1/2} &= \{F^e\}^{n+1/2}\end{aligned} \tag{3.137}$$

Multiplying equation 3.136 times Δt and $[M]$ we obtain,

$$[M^e]\{T^e\}^{n+1} - [M^e]\{T^e\}^{n+1/2} = [M^e]\frac{\Delta t}{2}\left\{\dot{T}^e\right\}^{n+1/2} + [M^e]\frac{\Delta t}{2}\left\{\dot{T}^e\right\}^{n+1} \tag{3.138}$$

Substituting equations 3.137 into equation 3.138 we obtain,

$$\begin{aligned}[M^e]\{T^e\}^{n+1} - [M^e]\{T^e\}^{n+1/2} =& \frac{\Delta t}{2}\{F^e\}^{n+1/2} - \frac{\Delta t}{2}[K^e]\{T^e\}^{n+1/2} \\ &+ \frac{\Delta t}{2}\{F^e\}^{n+1} - \frac{\Delta t}{2}[K^e]\{T^e\}^{n+1}\end{aligned} \tag{3.139}$$

Rearranging our equation such that terms with $\{T^e\}^{n+1}$ are the only terms on the left hand side of the equation we have,

$$\begin{aligned}\left[[M^e] + \frac{\Delta t}{2}[K^e]\right]\{T^e\}^{n+1} =& \left[[M^e] - \frac{\Delta t}{2}[K^e]\right]\{T^e\}^{n+1/2} \\ &+ \frac{\Delta t}{2}\left\{\{F^e\}^{n+1/2} + \{F^e\}^{n+1}\right\}\end{aligned} \tag{3.140}$$

Where our matrix equations may be written as,

$$M^e_{ij} = \int_{x_l}^{x_r} \psi^e_i \psi^e_j dx \tag{3.141}$$

$$K^e_{ij} = \int_{x_l}^{x_r} \kappa \frac{\partial \psi^e_i}{\partial x}\frac{\partial \psi^e_j}{\partial x} dx \tag{3.142}$$

Fractional Step Solution

Our fractional step solution to our convection diffusion problem consists of a half step through time using equations 3.126 -3.129 followed by a half step through the diffusion using equations 3.140 - 3.142. Our matrix equations have already been calculate with the exception of the mass matrix. We shall focus on the mass matrix at this point. The complete expansion of the mass matrix gives,

$$M^e_{ij} = \int_{x_l}^{x_r} \psi^e_i \psi^e_j dx = \int_{-1}^{+1} \psi^e_i \psi^e_j \frac{dx}{d\zeta} d\zeta = \frac{l^e}{2}\int_{-1}^{+1} \psi^e_i \psi^e_j d\zeta \tag{3.143}$$

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$$\begin{aligned}
M^e_{11} &= \frac{l^e}{2}\int_{-1}^{+1}\left[\frac{1}{2}(1-\zeta)\right]\left[\frac{1}{2}(1-\zeta)\right]d\zeta = \frac{2l^e}{6}\\
M^e_{12} &= \frac{l^e}{2}\int_{-1}^{+1}\left[\frac{1}{2}(1-\zeta)\right]\left[\frac{1}{2}(1+\zeta)\right]d\zeta = \frac{l^e}{6}\\
M^e_{21} &= \frac{l^e}{2}\int_{-1}^{+1}\left[\frac{1}{2}(1+\zeta)\right]\left[\frac{1}{2}(1-\zeta)\right]d\zeta = \frac{l^e}{6}\\
M^e_{22} &= \frac{l^e}{2}\int_{-1}^{+1}\left[\frac{1}{2}(1+\zeta)\right]\left[\frac{1}{2}(1+\zeta)\right]d\zeta = \frac{2l^e}{6}
\end{aligned} \tag{3.144}$$

The mass matrix becomes,

$$[M^e] = \frac{l^e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{3.145}$$

After the entire system of elements are assembled we wish to step our solution through time to obtain a steady state solution.

Given our temperature at any given point in time, T^n_k, our solution at the next time step follows the procedure,

- Solve the intermediate temperature at time step $n + 1/2$ from,

$$[\mathcal{M}]\{T\}^{n+1/2} = [\mathcal{M}]\{T\}^n - [\mathcal{C}]\{T\}^n - [\mathcal{D}]\{T\}^n \tag{3.146}$$

Where,

$$[\mathcal{M}^e] = \frac{l^e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{3.147}$$

$$[\mathcal{C}^e] = \frac{u\Delta t}{2}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \tag{3.148}$$

$$[\mathcal{D}^e] = \frac{(\Delta t)^2 u^2}{2l^e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{3.149}$$

Subject to the boundary conditions on the temperature.

- Using this intermediate temperature we advance our solution through the diffusion step.

$$\begin{aligned}
\left[[M] + \frac{\Delta t}{2}[K]\right]\{T\}^{n+1} = &\left[[M] - \frac{\Delta t}{2}[K]\right]\{T\}^{n+1/2}\\
&+\frac{\Delta t}{2}\left\{\{F\}^{n+1/2} + \{F\}^{n+1}\right\}
\end{aligned} \tag{3.150}$$

Where,

$$[M^e] = \frac{l^e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{3.151}$$

$$[K^e] = \frac{\kappa}{l^e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{3.152}$$

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Notice our forcing vector $\{F\}$ does not appear in either of our equations above. In the convection step this is due to the specification of the natural boundary condition. In this case everywhere that the temperature is not specified, its derivative is zero. For the diffusion step, we know already our boundary condition is a specified temperature at both ends of our domain. However, we have included the right hand side term in this diffusive step for completeness only.

Using this method we develop a Matlab code capable of producing a solution to our convection diffusion problem. A typical solutions are shown in figure 3.23. Notice that our numerical solution does not exactly match the exact solution at the nodes. At first this may seem a bit disturbing since we have already shown this is possible for the steady state case. However, this exact match is not possible from two and three dimensional problems anyway. As a result this method will be used to in our two dimensional problems. Accuracy of a numerical solution relates to how well a solution matches the exact solution. To this point this has been our focus. However there exist another issue of importance, stability. We have mentioned stability to some degree, equation 3.93 and figure 3.15. However, we have not formalized this discussion yet. In time dependent formulations, like those of this section, have a formalized approach to determining stability. A typical unstable solution using our fractional step methods is shown at the bottom of figure 3.23. Notice the oscillation of our numerical solution about the exact solution. We will generate a stability analysis method which may be used in advance to determine the stability of any matrix based numerical analysis method. We begin by noticing that in all of our time dependent numerical analysis methods our solution progresses through a time step is the same.

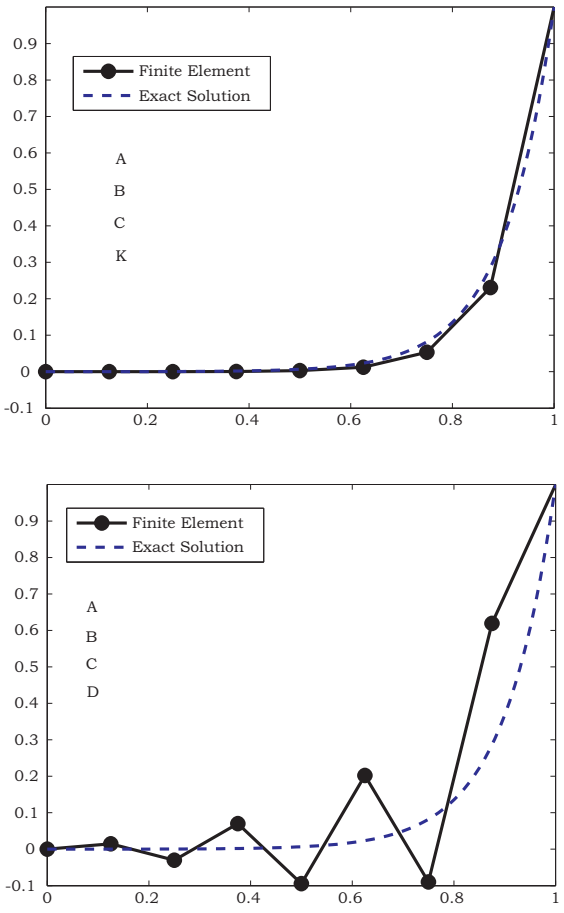


Figure 3.23: Fractional Step Solution

$$\{T\}^{n+1} = [G]\{T\}^{n} \tag{3.153}$$

As an example, the Lax-Wendoff method our amplification matrix, $[G]$, is,

$$[G] = [\mathcal{M}]^{-1}[[\mathcal{M}] - [\mathcal{C}] - [\mathcal{D}]] \tag{3.154}$$

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And the crank Nicholson method's amplification matrix is,

$$[G] = \left[[M] + \frac{\Delta t}{2}[K]\right]^{-1}\left[[M] - \frac{\Delta t}{2}[K]\right] \tag{3.155}$$

To determine the stability of any method we may take our nodal temperature values write them as,

$$\{T\}^k = \{T\}_e^k + \{\delta\}^k \tag{3.156}$$

Where $\{T\}_e^k$ is the solution to our equations. And $\{\delta\}^k$ is an error to our solution. Placing this representation into equation 3.153 we have,

$$\{T\}_e^{n+1} + \{\delta\}^{n+1} = [G]\{\{T\}_e^n + \{\delta\}^n\} \tag{3.157}$$

The boundary conditions on temperature are such that $\{\delta\}$ must be zero at every point in which $\{T\}$ is specified. As a result we may conclude that our error term may be broken away from our matrix and $\{\delta\}$ *should* be zero everywhere. However, lets express this equation as,

$$\{\delta\}^{n+1} = [G]\{\delta\}^n \tag{3.158}$$

Any initial value given for $\{\delta\}$ should reduce itself down to zero as time progresses. As stated above, the essential boundary conditions on $\{\delta\}$ are zero everywhere $\{T\}$ is specified. We now focus on our Amplification matrix $[G]$ and express look at a eigenvalues of this system. Obtaining the eigenvectors of this system we may write this matrix as,

$$[G] = [\mathcal{V}]^{-1}[\Lambda][\mathcal{V}] \tag{3.159}$$

Where $[\mathcal{V}]$ is a matrix consisting of columns of eigenvectors. Where $[\Lambda]$ is a diagonal matrix with its diagonal entries being the eigenvalues of the system matrix $[G]$. Substituting equation 3.159 into 3.158 we have,

$$\{\delta\}^{n+1} = [\mathcal{V}]^{-1}[\Lambda][\mathcal{V}]\{\delta\}^n \tag{3.160}$$

Multiplying both sides by $[\mathcal{V}]$ we obtain,

$$[\mathcal{V}]\{\delta\}^{n+1} = [\Lambda][\mathcal{V}]\{\delta\}^n \tag{3.161}$$

We may use a transformation,

$$\{\mathcal{R}\} = [\mathcal{V}]\{\delta\} \tag{3.162}$$

To rewrite our equations as,

$$\{\mathcal{R}\}^{n+1} = [\Lambda]\{\mathcal{R}\}^n \tag{3.163}$$

Since the matrix $[\Lambda]$ is a diagonal matrix with its diagonal entries being the eigenvalues we may write this system in index forma as,

$$\mathcal{R}_j^{n+1} = \lambda_j \mathcal{R}_j^n \tag{3.164}$$

If we stop and think about this equation we may immediately determine what could be thought of as a stability requirement. If the absolute value of any eigenvalue is less that unity then by sequentially multiplying our terms our, this error $\mathcal{R}_j$ will reduce as we progress our solution. For example, lets say we went thought 20 times steps, our equation would become,

$$\mathcal{R}_j^{n+20} = (\lambda_j)^{20}\mathcal{R}_j^n \tag{3.165}$$

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If the absolute value of our eigenvalue, λ_j, is less than unity, the error at step 20 is less than our error at our first step. In mathematical terms this may be written as,

$$\left|\mathcal{R}_j^{n+20}\right| < \left|\mathcal{R}_j^{n}\right| \quad \text{if} \quad = |\lambda_j| < 1 \tag{3.166}$$

For a stable numerical scheme to exist the magnitude of the eigenvalues, λ, of the amplification matrix $[G]$ must all be less than one,

$$|\lambda_{max}| \leq 1 \tag{3.167}$$

Interested reader are encouraged to read the text of Tannehill, Anderson, Fletcher [**?**] and the text of Hirsch [**?**]. If we look at our solutions in figure 3.23 we notice the maximum eigenvalues are listed in the plot. As we expect the eigenvalues over unity give solutions that are unstable. This instability originates from the Lax-Wendroff method in the fractional step method. The Crank-Nicholson method is unconditionally stable, in other words the method is stable regardless of the time step value used. Experimenting with the Matlab code one may obtain a rough relationship between the time step used in the Lax-Wendroff and the maximum eigenvalue of the matrix.

Fractional Step Matlab Code

The Matlab code that produced our figures for this section is shown and described in this section. The first portion of the listing is shown in figure 3.24. In this listing lines 1 to 10 simply describe the code, clear the variables and set the number of elements. Line 11 determines the length of each element. Line 13 specifies the conductivity, κ and line 15 specifies the velocity. Lines 16 and 17 calculates P, used later in the exact solution, and β used as a reference relative to the steady state solutions previously. Lines 18 and 19 are new and specify the Δt variable and the final time.

```
 1 %% Finite Element of 1-D Convective Diffusion (Copyright Dr R. Fithen)
 2 %  Filename: FEM1DConvFracStep.m
 3 % Uses Fractional Step Method
 4 %      Lax-Wendroff Step
 5 %         Followed by
 6 %      Crank-Nicholson Step
 7     clear
 8 % Number of elements
 9    Nelem=8;
10 % Length of each element
11     le=1.0/Nelem;
12 %conductivity
13    kappa=1.0;
14 % Specify u & Beta
15     u=10;
16     beta=u*le/2
17     P=u/kappa;
18     Dt=0.0001;
19    Tfinal=5;
```

Figure 3.24: Fractional Step Section #1

In section 2 of the listing the element matrices are constructed. Line 21 specifies the element level mass matrix equation 3.147 and

```
20 %Element Matrices.....
21     Me=le/6*[2 1;1 2];
22     Ce=u*Dt/2*[-1 1;-1 1];
23     De=Dt^2*u^2/2/le*[1 -1;-1 1];
24     Ke=kappa/le*[1 -1;-1 1];
25 % connectivity matrix...
26     conn(:,1)=1:Nelem;
27     conn(:,2)=2:Nelem+1;
```

Figure 3.25: Fractional Step Section #2

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3.151. Line 22 specifies the convection matrix, equation 3.148. Lines 23 and 24 specify the diffusive matrices, equation 3.149 and 3.152. Lines 26 and 27 specify the connectivity exactly the same way we have done in all of our previous Matlab codes.

```
28 %Build global matrix
29     Kcg=zeros(Nelem+1,Nelem+1);
30     Kdg=zeros(Nelem+1,Nelem+1);
31     Fcgm=zeros(Nelem+1,Nelem+1); % Matrix to construct vector (convection)
32     Fdgm=zeros(Nelem+1,Nelem+1); % matrix to construct vector (diffusion)
33      for i=1:Nelem
34         Kcg(conn(i,:),conn(i,:))=Kcg(conn(i,:),conn(i,:))+Me;
35         Kdg(conn(i,:),conn(i,:))=Kdg(conn(i,:),conn(i,:))+Me+Dt/2*Ke;
36         Fcgm(conn(i,:),conn(i,:))=Fcgm(conn(i,:),conn(i,:))+Me-Ce-De;
37         Fdgm(conn(i,:),conn(i,:))=Fdgm(conn(i,:),conn(i,:))+Me-Dt/2*Ke;
38      end
```

Figure 3.26: Fractional Step Section #3

Lines 29 to 32 create our matrices for both the Lax-Wendroff and our Crank-Nicholson methods. Line 34 constructs the left hand side matrix of our Lax-Wendroff method, $[\mathcal{M}]$ in equations 3.146 and 3.147. Line 35 constructs the left hand side matrix of our Crank-Nicholson method, equations 3.150 and 3.151-3.152. Line 36 calculates the right hand side matrix for the Lax-Wendroff method. This matrix is a complete form that is multiplied by the temperature from the previous step, $\{T\}^n$. This matrix is shown in equations 3.146 to 3.149. Line 37 calculates the right hand side matrix associated with the Crank-Nicholson method. This matrix is shown in equations 3.150 to 3.152.

In section 4 of the listing, lines 40 and 41 we specify the initial temperature as zero throughout the domain and one on the right end of the domain. These initial conditions are consistent with the boundary conditions of the problem. Lines 43 and 44 prepare the matrix for specifying the boundary conditions. This matrix is actually the mass matrix, equation

```
39 % Inital temperature
40     Tn=zeros(Nelem+1,1);
41     Tn(Nelem+1,1)=1;
42 % Place Boundary Conditions
43     Kcg(1,:)=0;
44     Kcg(1,1)=1;
45     Kcg(Nelem+1,:)=0;
46     Kcg(Nelem+1,Nelem+1)=1;
47     Kdg(1,:)=0;
48     Kdg(1,1)=1;
49     Kdg(Nelem+1,:)=0;
50     Kdg(Nelem+1,Nelem+1)=1;
```

Figure 3.27: Fractional Step Section #4

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Problem 1

Given the problem,

$$\frac{\partial^2 T}{\partial x^2} + 5x^4 = 0$$

In the domain $1 \leq x \leq 2$. Subject to the boundary conditions,

$$T\left(x = 1\right) = 0$$

and

$$\left.\frac{\partial T}{\partial x}\right|_{x=2} = 0$$

Using the approximation,

$$T_a\left(x\right) = \sum_{i=1}^{N} c_i \psi_i\left(x\right)$$

Where,

$$\psi_i\left(x\right) = \cos\left[\left(x - 2\right)\frac{\left(2i - 1\right)\pi}{2}\right]$$

Find the following weighted residual solutions and determine the solution norms for:

- Galerkin.
- Petrov-Galerkin with the weighting function, $w_k\left(x\right) = x^k$.
- Collocation method using Tchebychev-Gauss-Labatto distribution with no end points.
- Least Squares.

Present the table of norms as well as several lower order solutions with the exact solution as a comparison.

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Chapter 4

Appendix A

4.0.1 Cylindrical Coordinates

For cylindrical coordinates the continuity equation may be written as,

$$\frac{1}{r}\frac{\partial\left(rv_r\right)}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{\partial v_z}{\partial z}=0 \tag{4.1}$$

Where

- v_r is the velocity in the r direction.
- v_θ is the velocity in the θ direction.
- v_z is the velocity in the z direction.

As a result the velocity may be written as,

$$\vec{V}=v_r\vec{e}_r+v_\theta\vec{e}_\theta+v_z\vec{e}_z \tag{4.2}$$

Where

- $\vec{e}_r$ is the base vector in the r direction, shown below.
- $\vec{e}_\theta$ is the base vector in the θ direction, shown below.
- $\vec{e}_z$ is the base vector in the z direction, shown below.

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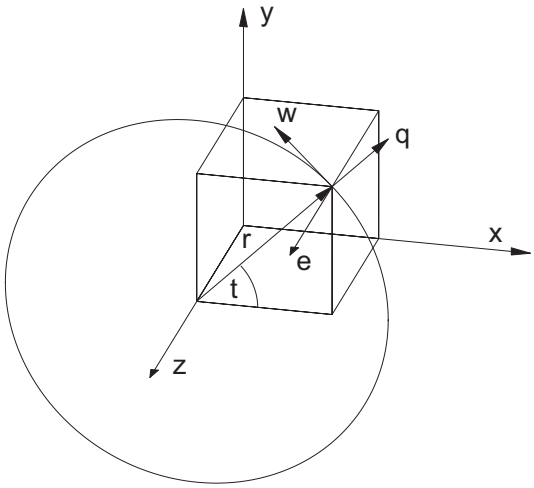


Figure 4.1: Base Vectors in Cylindrical Coordinates

4.0.2 Cylindrical Coordinates

In cylindrical coordinates these equations may be written as,

(r-momentum)

$$\rho\left[\frac{\partial v_r}{\partial t}+ v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}+v_z\frac{\partial v_r}{\partial z}-\frac{v_\theta^2}{r}\right] = -\frac{\partial p}{\partial r}+\rho g_r+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial^2 v_r}{\partial z^2}-\frac{v_r}{r^2}\right] \tag{4.3}$$

(θ-momentum)

$$\rho\left[\frac{\partial v_\theta}{\partial t}+ v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+v_z\frac{\partial v_\theta}{\partial z}+\frac{v_\theta v_r}{r}\right] = -\frac{1}{r}\frac{\partial p}{\partial \theta}+\rho g_\theta+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}+\frac{\partial^2 v_\theta}{\partial z^2}-\frac{v_\theta}{r^2}\right] \tag{4.4}$$

(z-momentum)

$$\rho\left[\frac{\partial v_z}{\partial t}+ v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right] = -\frac{\partial p}{\partial z}+\rho g_z+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right] \tag{4.5}$$

4.0.3 Cylindrical Coordinates

$$\rho c_p\left[\frac{\partial T}{\partial t}+ v_r\frac{\partial T}{\partial r}+\frac{v_\theta}{r}\frac{\partial T}{\partial \theta}+v_z\frac{\partial T}{\partial z}\right] = \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}+\frac{\partial^2 T}{\partial z^2}\right]+\mu\Phi+\dot{q} \tag{4.6}$$

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Where the viscous dissipation is,

$$\Phi = \left[\left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z}\right)^2 + \left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\right)^2 + 2\left[\left(\frac{\partial v_r}{\partial r}\right)^2 + \left(\frac{\partial v_z}{\partial z}\right)^2 + \left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_r}{\partial r}\right)^2\right]\right] \tag{4.7}$$

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