

Mathematics of the FEM

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Abstract

In this course we will use Maxima to investigate a few concepts leading to the development of the Finite Element Method. We will,

- Investigate the general Rayleigh Ritz weighted residual formulation.
- Use Maxima to develop the equations for a 1-D second order equation.
- Obtain a general solution to our 2nd order equation.
- Investigate convergence of the weighted residual methods.
- Convert our continuous weighted residual formulation to a piecewise continuous formulation, (I.E. Finite Element Method)
- Develop the concepts of assemble of our Finite Element equations.
- Set the stage for the next course, Finite Elements of Beams and Bars!

1. Model Problem

Our Model Problem may be written as,

$$k\frac{d^2T}{dx^2} + S(x) = 0 \tag{1}$$

Where the term $S(x)$, is an arbitrary function of x. We shall also take our boundary conditions as,

$$T(x=0) = T(x=1) = 0 \tag{2}$$

For our problem we will take our source function $S(x)$ as,

$$S(x) = -25\sin(4\pi x) - 33\sin(3\pi x) \tag{3}$$

using this source function the exact solution to this differential equation is,

$$T_e = -\frac{25}{16\pi^2 k}\sin(4\pi x) - \frac{11}{3\pi^2 k}\sin(3\pi x) \tag{4}$$

To check this solution refer to Maxima File $\Longrightarrow$ $\Longrightarrow$ STOP REVIEW $\Longrightarrow$ PDH2011_01.wxm

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Lets assume a solution of the form,

$$T_a(x) = a_0 + a_1 x + a_2 x^2 \tag{5}$$

Where a_0, a_1, and a_2 are constants, yet to be determined. The notation T_a is used to denote an approximation, where T_e is the exact solution. Since our solution is only approximate inserting it in our original equation will give and error, or residual term ($\mathcal{R}$),

$$\mathcal{R}(x, a_0, a_1, a_2) = k\frac{d^2T_a^e}{dx^2} + S \tag{6}$$

if we inserted our equation for T_a into our residual equation we obtain,

$$\mathcal{R}(x, a_0, a_1, a_2) = a_2 2k - 25\sin(4\,\pi\,x) - 33\,\sin(3\,\pi\,x) \tag{7}$$

There are two question that should enter our minds at this time,

1. What happened to the constants a_0 and a_1. And, how do we find these constants?
2. From our residual equation how can we determine a_2?

First we notice the boundary conditions in Equation 2 above. By applying these 2 equation directly to our assumed solution, equation 10 we obtain,

$$T_a(x=0) = a_0 = 0 \tag{8}$$

and

$$T_a(x=1) = a_1 + a_2 = 0 \tag{9}$$

Which given the second order approximation as,

$$T_a(x) = a_1 x(1-x) \tag{10}$$

Inserting this approximation into our residual we get,

$$\mathcal{R}(x, a_1) = -a_1 2k - 25\sin(4\pi x) - 33\sin(3\pi x) \tag{11}$$

We notice that our residual is a function of both x and a_1. Hence we have One equation and Two unknowns! However, weighted residuals come to the rescue! We may specify and arbitrary function $w(x)$ and force the following to be true,

$$\int_0^1 w(x)\,\mathcal{R}(x, a_1)\,dx = 0 \tag{12}$$

The integration will KNOCK out the x dependency of this equation such that it can be written as,

$$\mathcal{E}(a_1) = \int_0^1 w(x)\,\mathcal{R}(x, a_1)\,dx = 0 \tag{13}$$

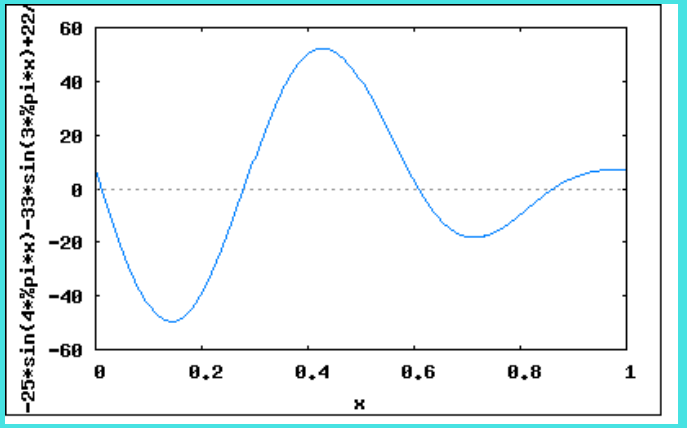
Lets take a function $w(x) = 1$ for our case. This means our residual mush have a zero area in the domain $0 < x < 1$. If we take our constant k to be a value of unity, $k = 1$ our equation becomes.

$$a_1 = -\frac{11}{\pi} \tag{14}$$

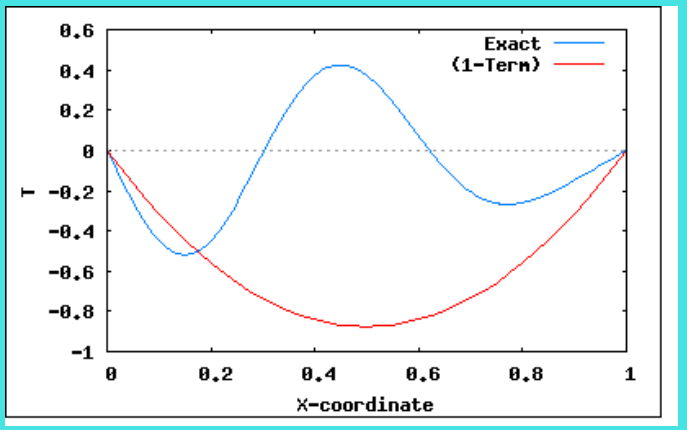
Our approximate solution becomes

$$T_a(x) = \frac{11x(x-1)}{\pi} \tag{15}$$

A plot of the residual is shown below, Area under the curve is the same as the area over the curve.



A plot of the exact solution along with our approximate solution is shown below.



NOTE: Blue is the exact and Red is our approximate

Now lets try to get a better, more accurate solution!
Lets assume a solution as,

$$T_a(x) = a_1 x(1-x) + a_2 x^2(1-x) \tag{16}$$

Notice our two functions,

$$x(1-x) \tag{17}$$

AND

$$x^2(1-x) \tag{18}$$

satisfy our boundary conditions, 2. Inserting our approximate solution into our residual equation,

$$\mathcal{R}(x, a_1, a_2) = -25\sin(4\pi x) - 33\sin(3\pi x) - 6xa_2 + 2a_2 - 2a_1 \tag{19}$$

Now we have One equation and Three unknowns, x, a_1, a_2. However, weighted residuals will solve this problem as well. We pick two different weighting functions, $w_1(x)$ and $w_2(x)$ and require the following two equations to be true,

$$\mathcal{E}_1(a_1, a_2) = \int_0^1 w_1(x)\,\mathcal{R}(x, a_1, a_2)\,dx = 0 \tag{20}$$

And

$$\mathcal{E}_2(a_1, a_2) = \int_0^1 w_2(x)\,\mathcal{R}(x, a_1, a_2)\,dx = 0 \tag{21}$$

For simplicity we will pick our two functions as,

$$w_1(x) = 1 \tag{22}$$

And

$$w_2(x) = x \tag{23}$$

Carrying out these integral we obtain, Maxima,

$$22 + \pi a_2 + 2\pi a_1 = 0 \tag{24}$$

And

$$19 + 4\pi a_2 + 4\pi a_1 = 0 \tag{25}$$

Maxima gives a solution as,

$$a_1 = -\frac{69}{4\pi} \tag{26}$$

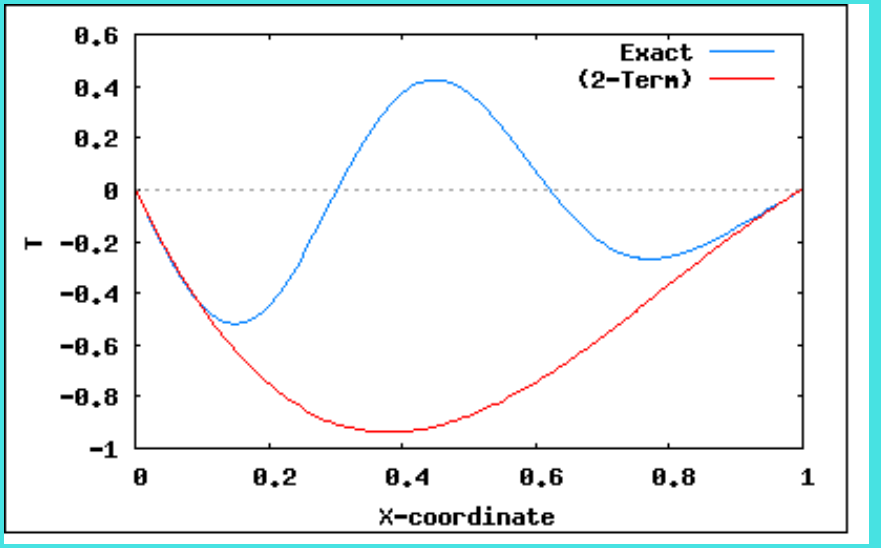
And

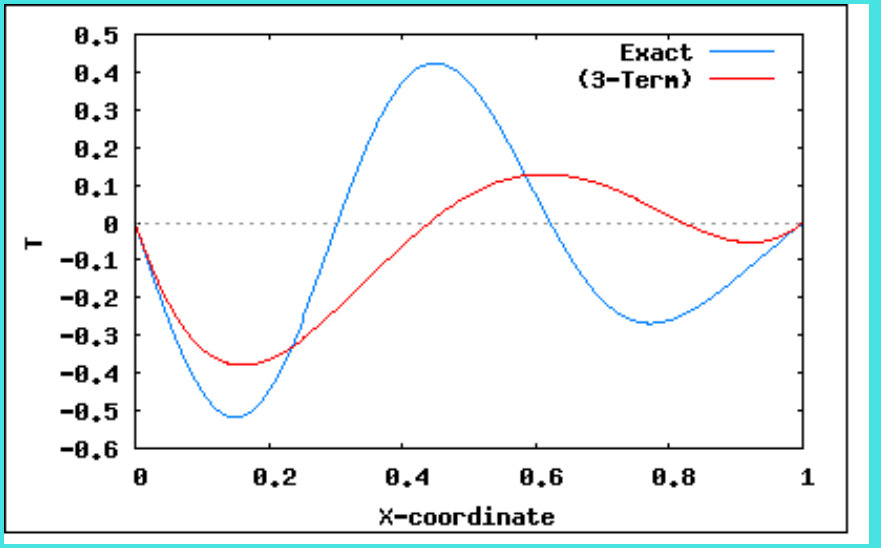
$$a_2 = \frac{25}{2\pi} \tag{27}$$

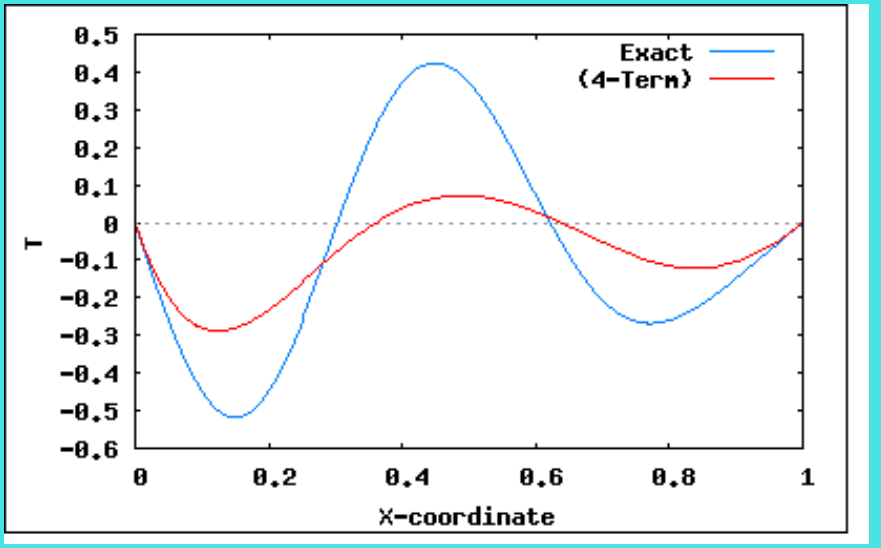
placing this result back into our equation gives,

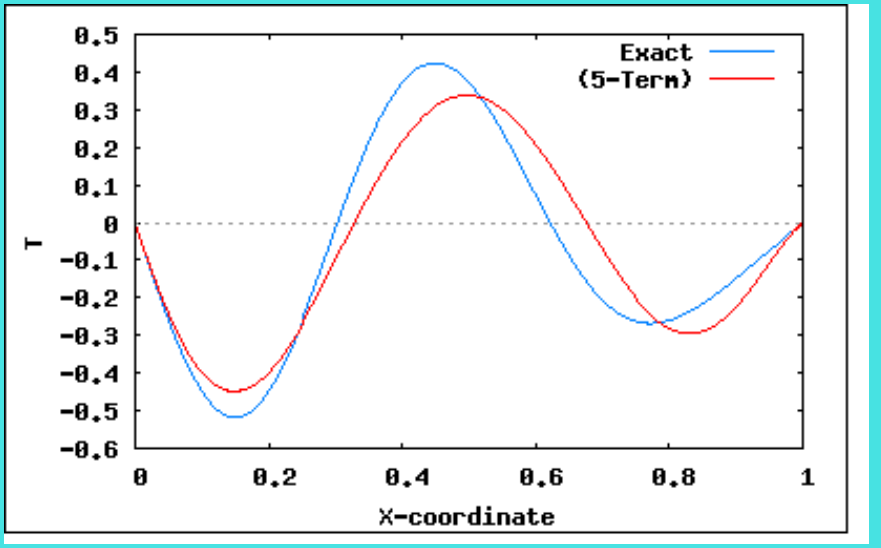
$$T_a(x) = -\frac{69}{4\pi} x(1-x) + \frac{25}{2\pi} x^2(1-x) \tag{28}$$

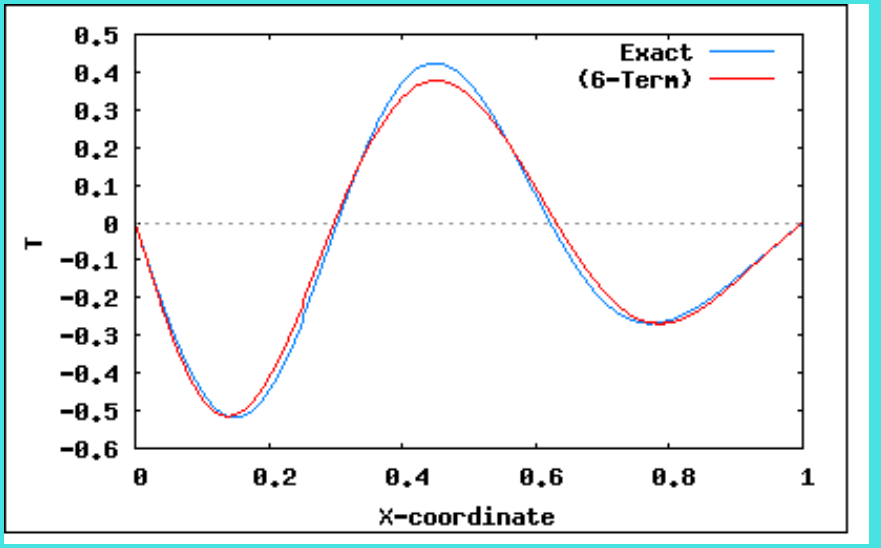
This solution is plotted on the next page. In addition several higher order solutions are included on later pages.

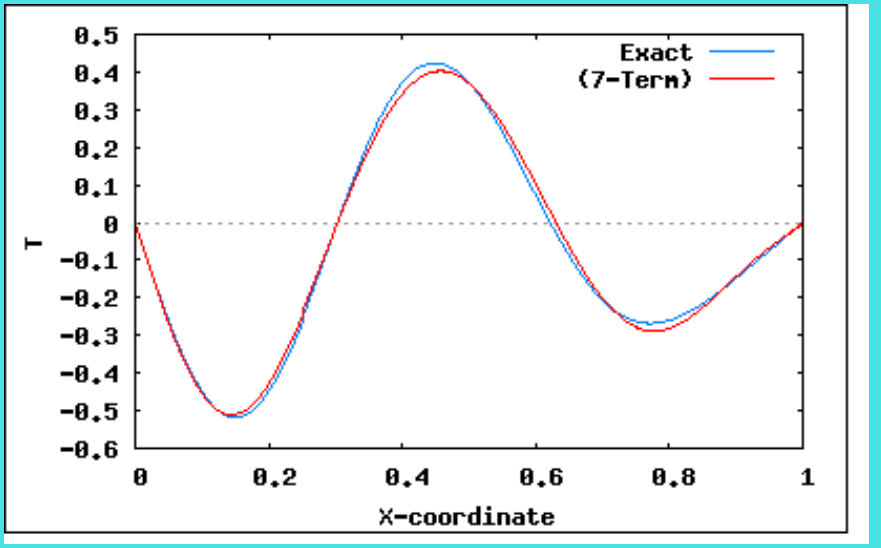

Exact
(2-Term)
T
X-coordinate
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
0.2
0.4
0.6
0.8
1

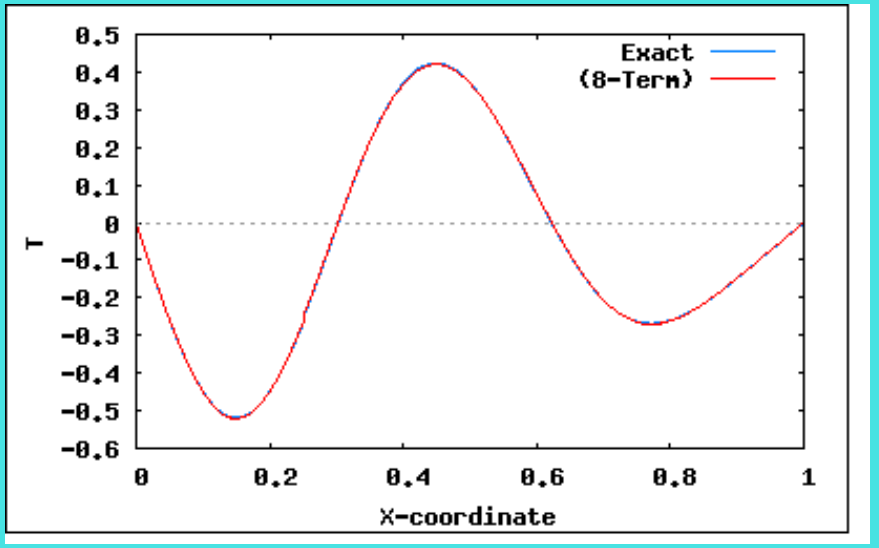

Exact
(3-Term)
T
X-coordinate
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
0
0.2
0.4
0.6
0.8
1


Exact
(4-Term)
T
X-coordinate
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
0
0.2
0.4
0.6
0.8
1


Exact
(5-Term)
T
X-coordinate
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
0
0.2
0.4
0.6
0.8
1


Exact
(6-Term)
T
X-coordinate
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
0
0.2
0.4
0.6
0.8
1


Exact
(7-Term)
T
X-coordinate
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
0
0.2
0.4
0.6
0.8
1



Notice how the weighted residual solution converges to the exact solution. In fact if we carried more and more term we would eventually obtain the exact solution. See Maxima files!

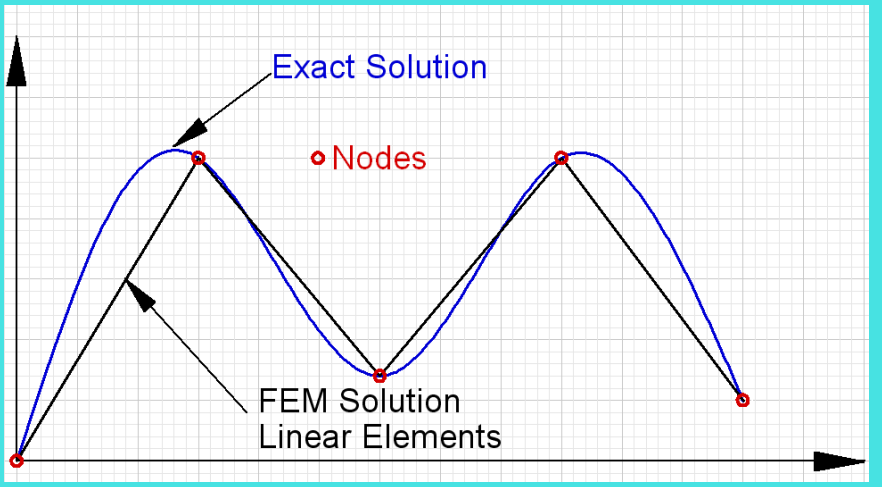
2. Finite Element Model

In the weighted residual we used functions that were continuous throughout the domain. In the Finite Element Method we seek a solution to our equation using piecewise functions.

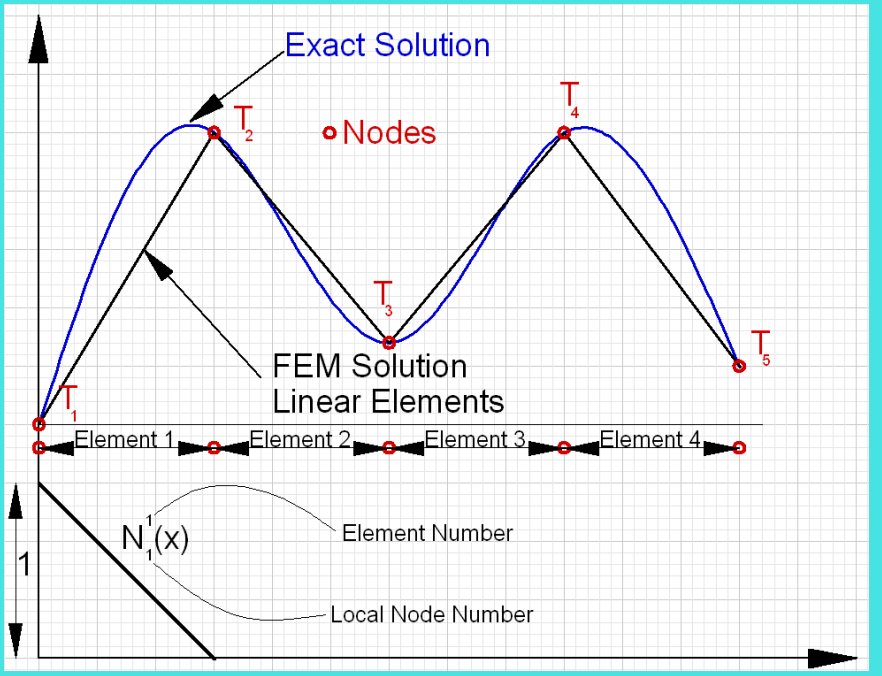
We will seek answers to the following:

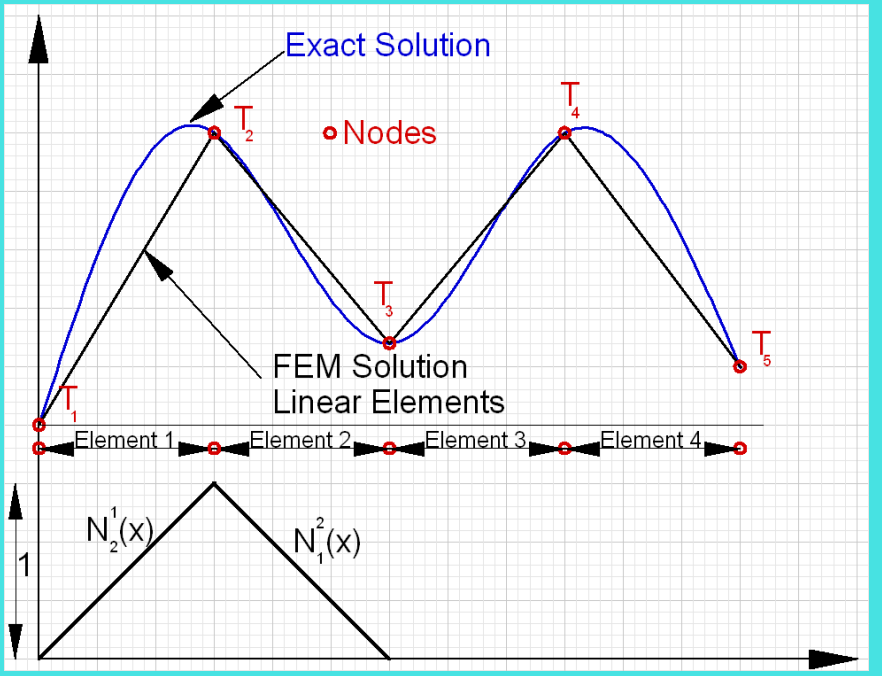
- How do we construct these functions?
- How do we combine our functions together?
- How do we solve the resulting system?

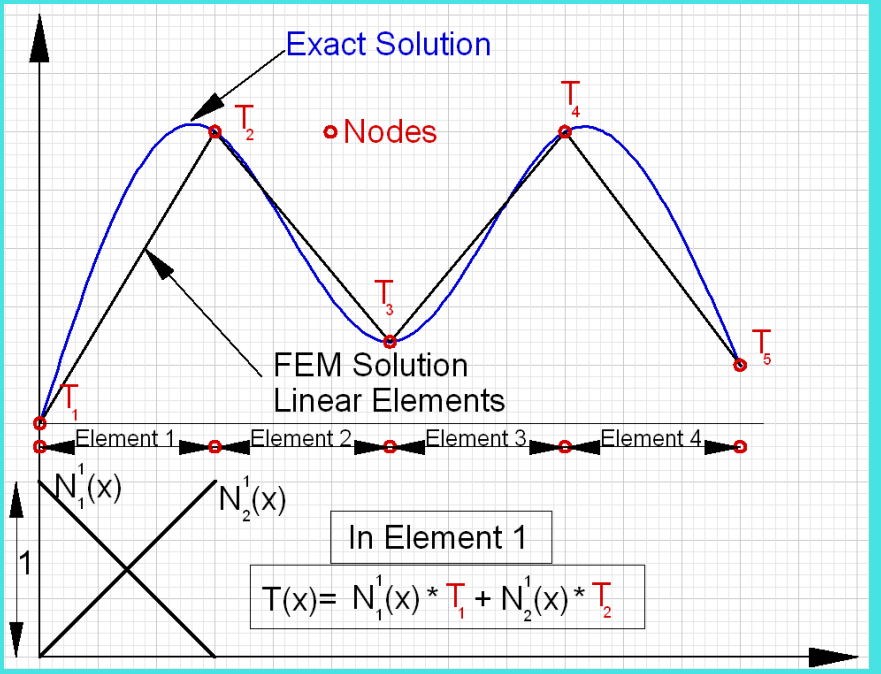
By observing the following plot we see a typical exact solution and a finite element approximation of this solution.



Unlike the weighted residual method the approximation functions are linear piecewise functions. The question is, How do we produce these functions? We know how to simply draw a straight line between two points. However, we must determine a functional representation of this in order to use the weighted residual methods.


Exact Solution
Nodes
T1
T2
T3
T4
T5
FEM Solution
Linear Elements
Element 1
Element 2
Element 3
Element 4
1
N1(1)(x)
Element Number
Local Node Number


Exact Solution
Nodes
T_1
T_2
T_3
T_4
T_5
FEM Solution
Linear Elements
Element 1
Element 2
Element 3
Element 4
1
$N_2^1(x)$
$N_1^2(x)$



Where the Function $N_N^e(x)$ may be written as,

$$N_1^e(x) = \left(1 - \frac{x}{L^e}\right) \tag{29}$$

$$N_2^e(x) = \frac{x}{L^e} \tag{30}$$

2.1. Weight Residual Statement

Remember we have a weighted residual statement,

$$\mathcal{E}_1\left(T_1^e, T_2^e\right) = \int_0^{L^e} w_1\left(x\right) \mathcal{R}\left(x, T_1^e, T_2^e\right) dx = 0 \tag{31}$$

And

$$\mathcal{E}_2\left(T_1^e, T_2^e\right) = \int_0^{L^e} w_2\left(x\right) \mathcal{R}\left(x, T_1^e, T_2^e\right) dx = 0 \tag{32}$$

For simplicity we shall ALWAYS use

$$w_1\left(x\right) = N_1^e\left(x\right) \tag{33}$$

$$w_2\left(x\right) = N_2^e\left(x\right) \tag{34}$$

From Equation 35 we shall write our residual as,

$$\mathcal{R}\left(x, T_1^e, T_2^e\right) = k\frac{d^2T_a^e}{dx^2} + S \tag{35}$$

Hence our 2 equations become!

$$\mathcal{E}_1\left(T_1^e, T_2^e\right) = \int_0^{L^e} N_1^e\left(x\right)\left[k\frac{d^2T_a^e}{dx^2} + S\right] dx = 0 \tag{36}$$

And

$$\mathcal{E}_2\left(T_1^e, T_2^e\right) = \int_0^{L^e} N_2^e\left(x\right)\left[k\frac{d^2T_a^e}{dx^2} + S\right] dx = 0 \tag{37}$$

In symbolic (index i on $\mathcal{E}$) form we have

$$\mathcal{E}_i\left(T_1^e, T_2^e\right) = \int_0^{L^e} N_i^e\left(x\right)\left[k\frac{d^2T_a^e}{dx^2} + S\right] dx = 0 \tag{38}$$

2.2. Integration by Parts

Remember this simple rule in Calculus,

$$d(uv) = udv + vdu \tag{39}$$

Rearranging,

$$udv = -vdu + d(uv) \tag{40}$$

If u and v are functions of x we can write,

$$u\frac{dv}{dx}dx = -v\frac{du}{dx}dx + d(uv) \tag{41}$$

We now integrate this function between $x = 0$ and $x = L^e$

$$\int_0^{L^e} u\frac{dv}{dx}dx = -\int_0^{L^e} v\frac{du}{dx}dx + \int_0^{L^e} d(uv) \tag{42}$$

Or

$$\int_0^{L^e} u\frac{dv}{dx}dx = -\int_0^{L^e} v\frac{du}{dx}dx + [uv]_0^{L^e} \tag{43}$$

Which is INTEGRATION BY PARTS. By noticing one term in our equation,

$$\int_0^{L^e} N_i^e(x)\, k\frac{d^2T_a^e}{dx^2}dx \tag{44}$$

We may use this integration by parts noticing,

$$u = N_i^e(x) \tag{45}$$

And

$$v = k\frac{dT_a^e}{dx} \tag{46}$$

Substituting we obtain,

$$\int_0^{L^e} N_i^e(x)\, k\frac{d^2T_a^e}{dx^2}dx = -\int_0^{L^e} \frac{dN_i^e(x)}{dx}k\frac{dT_a^e}{dx}dx + \left[N_i^e(x)\, k\frac{dT^e}{dx}\right]_0^{L^e} \tag{47}$$

Expressing ALL terms we have,,

$$\int_0^{L^e} N_i^e(x)\left[k\frac{d^2T_a^e}{dx^2}+S(x)\right]dx=\int_0^{L^e}\left[N_i^e(x)S(x)-\frac{dN_i^e(x)}{dx}k\frac{dT_a^e}{dx}\right]dx+\left[N_i^e(x)\,k\frac{dT^e}{dx}\right]_0^{L^e} \tag{48}$$

Heat Transfer Equations,

$$\frac{d}{dx}\left[k\frac{dT}{dx}\right]+S(x)=0 \tag{49}$$

Bar Equation,

$$\frac{d}{dx}\left[AE\frac{dU}{dx}\right]+S(x)=0 \tag{50}$$

We Will now Convert our notation to the BAR equation..

$$\int_0^{L^e} N_i^e(x)\left[AE\frac{d^2U_a^e}{dx^2}+S(x)\right]dx=\int_0^{L^e}\left[N_i^e(x)S(x)-\frac{dN_i^e(x)}{dx}AE\frac{dU_a^e}{dx}\right]dx+\left[N_i^e(x)\underbrace{AE\frac{dU^e}{dx}}_{\text{BELOW}}\right]_0^{L^e} \tag{51}$$

Notice the last term in this equation,

$$\left[AE\frac{dU^e}{dx}\right]_{x=L^e}=P \tag{52}$$

Where P is the applied load. NOTE:

$$\epsilon=\frac{dU^e}{dx} \tag{53}$$

and

$$\sigma=E\epsilon \tag{54}$$

Therefore,

$$\frac{P}{A}=\sigma=E\frac{dU^e}{dx} \tag{55}$$

Our Element equation now becomes,

$$\int_0^{L^e}\left[N_i^e(x)S(x)-\frac{dN_i^e(x)}{dx}AE\frac{dU_a^e}{dx}\right]dx+\left[N_i^e(x)P\right]_0^{L^e} \tag{56}$$

Remember we have 2 equations,

$$\mathcal{E}_1^e = \int_0^{L^e} \left[N_1^e(x) S(x) - \frac{dN_1^e(x)}{dx} AE \frac{dU_a^e}{dx} \right] dx + \left[N_1^e(x) P \right]_0^{L^e} \tag{57}$$

$$\mathcal{E}_2^e = \int_0^{L^e} \left[N_2^e(x) S(x) - \frac{dN_2^e(x)}{dx} AE \frac{dU_a^e}{dx} \right] dx + \left[N_2^e(x) P \right]_0^{L^e} \tag{58}$$

Or

$$\mathcal{E}_1^e = \int_0^{L^e} \left[N_1^e(x) S(x) - \frac{dN_1^e(x)}{dx} AE \frac{dU_a^e}{dx} \right] dx + \underbrace{N_1^e(L^e)}_{=0} P_{L^e} - \underbrace{N_1^e(0)}_{=1} P_0 \tag{59}$$

$$\mathcal{E}_2^e = \int_0^{L^e} \left[N_2^e(x) S(x) - \frac{dN_2^e(x)}{dx} AE \frac{dU_a^e}{dx} \right] dx + \underbrace{N_2^e(L^e)}_{=1} P_{L^e} - \underbrace{N_2^e(0)}_{=0} P_0 \tag{60}$$

Note our equation are labeled, $\mathcal{E}_n^e$ Where e is the element number and n is the local node number. In our case there are 2 local node numbers and we may chose the number of element we wish, more gives better answers.

$$\mathcal{E}_1^e = \int_0^{L^e} \left[N_1^e(x) S(x) - \frac{dN_1^e(x)}{dx} AE \frac{dU_a^e}{dx} \right] dx - P_l^e \tag{61}$$

$$\mathcal{E}_2^e = \int_0^{L^e} \left[N_2^e(x) S(x) - \frac{dN_2^e(x)}{dx} AE \frac{dU_a^e}{dx} \right] dx + P_r^e \tag{62}$$

Notice the notation change from $P_0 \Longrightarrow P_l^e$ and $P_{L^e} \Longrightarrow P_r^e$ the subscripts denote the left and right ends of the element. Our displacement U may be written as,

$$U_a^e = N_1^e(x) U_1^e + N_2^e(x) U_2^e = \sum_{j=1}^{2} N_j^e(x) U_j^e \tag{63}$$

Inserting this equation back into our expressions,

$$\mathcal{E}_1^e = \left[\int_0^{L^e} N_1^e(x) S(x) dx - P_l^e \right] - \sum_{j=1}^{2} \left[\int_0^{L^e} \frac{dN_1^e(x)}{dx} AE \frac{dN_j^e(x)}{dx} dx \right] U_j^e \tag{64}$$

$$\mathcal{E}_2^e = \left[\int_0^{L^e} N_2^e(x)\, S(x)\, dx + P_r^e\right] - \sum_{j=1}^{2}\left[\int_0^{L^e} \frac{dN_2^e(x)}{dx} AE \frac{dN_j^e(x)}{dx} dx\right] U_j^e \tag{65}$$

Lets focus on our terms with the summation, rewriting our equations,

$$\mathcal{E}_1^e = \mathcal{F}_1^e - \sum_{j=1}^{2}\left[\int_0^{L^e} \frac{dN_1^e(x)}{dx} AE \frac{dN_j^e(x)}{dx} dx\right] U_j^e \tag{66}$$

$$\mathcal{E}_2^e = \mathcal{F}_2^e - \sum_{j=1}^{2}\left[\int_0^{L^e} \frac{dN_2^e(x)}{dx} AE \frac{dN_j^e(x)}{dx} dx\right] U_j^e \tag{67}$$

Where

$$\mathcal{F}_1^e = \left[\int_0^{L^e} N_1^e(x)\, S(x)\, dx - P_l^e\right] \tag{68}$$

$$\mathcal{F}_2^e = \left[\int_0^{L^e} N_2^e(x)\, S(x)\, dx + P_r^e\right] \tag{69}$$

Expanding our summation out,

$$\mathcal{E}_1^e = \mathcal{F}_1^e - \left[\int_0^{L^e} \frac{dN_1^e(x)}{dx} AE \frac{dN_1^e(x)}{dx} dx\right] U_1^e + \left[\int_0^{L^e} \frac{dN_1^e(x)}{dx} AE \frac{dN_2^e(x)}{dx} dx\right] U_2^e \tag{70}$$

$$\mathcal{E}_2^e = \mathcal{F}_2^e - \left[\int_0^{L^e} \frac{dN_2^e(x)}{dx} AE \frac{dN_1^e(x)}{dx} dx\right] U_1^e + \left[\int_0^{L^e} \frac{dN_2^e(x)}{dx} AE \frac{dN_2^e(x)}{dx} dx\right] U_2^e \tag{71}$$

Or,

$$\mathcal{E}_1^e = \mathcal{F}_1^e - K_{11}^e U_1^e + K_{12}^e U_2^e \tag{72}$$

$$\mathcal{E}_2^e = \mathcal{F}_2^e - K_{21}^e U_1^e + K_{22}^e U_2^e \tag{73}$$

Interesting, this looks a lot like a matrix equation, However, we must add these equation for ALL elements! Lets say we have 4 elements. Our weighted residual statement will be,

$$\sum_{e=1}^{4}\sum_{i=1}^{2} \mathcal{E}_i^e = 0 \tag{74}$$

Before we solve our system of equation we must first determine

$$K_{ij}^{e} = \int_{0}^{L^{e}} \frac{dN_{i}^{e}(x)}{dx} AE \frac{dN_{j}^{e}(x)}{dx} dx \tag{75}$$

If our area A, and modulus E, is constant then

$$K_{ij}^{e} = AE \int_{0}^{L^{e}} \frac{dN_{i}^{e}(x)}{dx} \frac{dN_{j}^{e}(x)}{dx} dx \tag{76}$$

Remembering our functions,

$$N_{1}^{e}(x) = \left(1 - \frac{x}{L^{e}}\right) \tag{77}$$

$$N_{2}^{e}(x) = \frac{x}{L^{e}} \tag{78}$$

We notice our derivatives are,

$$\frac{dN_{1}^{e}}{dx} = -\frac{1}{L^{e}} \tag{79}$$

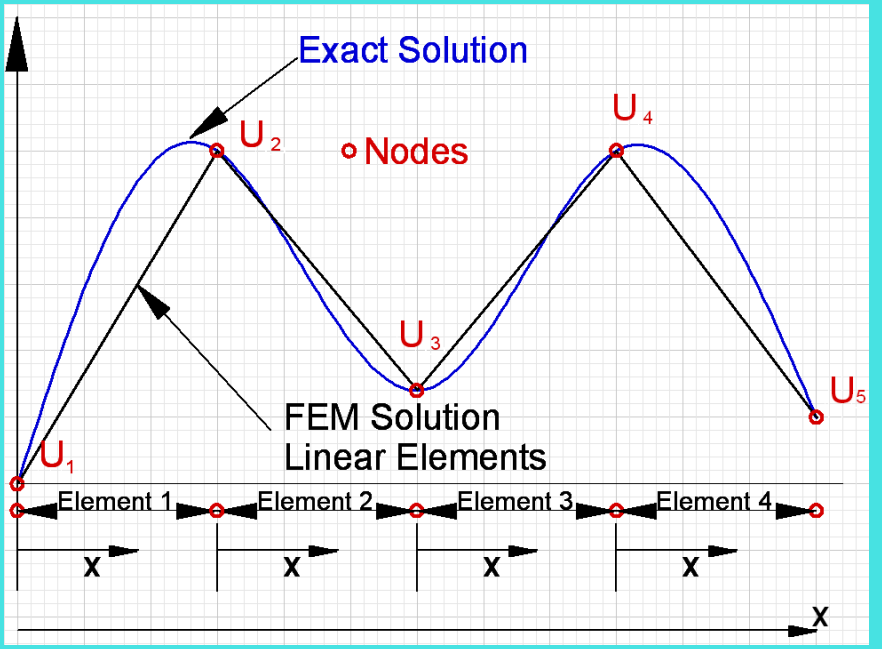
$$\frac{dN_{2}^{e}}{dx} = \frac{1}{L^{e}} \tag{80}$$

Therefore our Matrix Becomes,

$$[K] = \frac{AE}{L^{e}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{81}$$

Before we begin assembling these equations there are several points to make,

- Each Element matrix is the same, only the Length of the element or the AE term will change the numerical value of the element matrix.
- A local coordinate system for each element will ensure easy integration of the matrix, see figure.
- Assembly will drop out inter-element boundary terms.



Looking back at equations 70 to 73 we notice of element matrix is of the form,

$$\frac{AE}{L^e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} U_1^e \\ U_2^e \end{Bmatrix} = \begin{Bmatrix} \mathcal{F}_1^e \\ \mathcal{F}_2^e \end{Bmatrix} \tag{82}$$

These equations must be summed or put together for all elements in the domain! HOW? We begin by noticing how the local element displacement corresponds to the global displacement. For example

element one may be written in two ways,

$$\frac{AE}{L^1}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}U_1^1\\ U_2^1\end{Bmatrix}=\begin{Bmatrix}\mathcal{F}_1^1\\ \mathcal{F}_2^1\end{Bmatrix} \tag{83}$$

OR

$$\frac{AE}{L}\begin{bmatrix}1 & -1 & 0 & 0 & 0\\ -1 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}\begin{Bmatrix}U_1\\ U_2\\ U_3\\ U_4\\ U_5\end{Bmatrix}=\begin{Bmatrix}\mathcal{F}_1^1\\ \mathcal{F}_2^1\\ 0\\ 0\\ 0\end{Bmatrix} \tag{84}$$

For the sake of convenience we will make all of our elements the same length $L^1 = L^2 = L^3 = L^4 = L$. Element 2 becomes,

$$\frac{AE}{L}\begin{bmatrix}0 & 0 & 0 & 0 & 0\\ 0 & 1 & -1 & 0 & 0\\ 0 & -1 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}\begin{Bmatrix}U_1\\ U_2\\ U_3\\ U_4\\ U_5\end{Bmatrix}=\begin{Bmatrix}0\\ \mathcal{F}_1^2\\ \mathcal{F}_2^2\\ 0\\ 0\end{Bmatrix} \tag{85}$$

Element 3,

$$\frac{AE}{L}\begin{bmatrix}0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & -1 & 0\\ 0 & 0 & -1 & 1 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}\begin{Bmatrix}U_1\\ U_2\\ U_3\\ U_4\\ U_5\end{Bmatrix}=\begin{Bmatrix}0\\ 0\\ \mathcal{F}_1^3\\ \mathcal{F}_2^3\\ 0\end{Bmatrix} \tag{86}$$

Element 4,

$$\frac{AE}{L}\begin{bmatrix}0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & -1\\ 0 & 0 & 0 & -1 & 1\end{bmatrix}\begin{Bmatrix}U_1\\ U_2\\ U_3\\ U_4\\ U_5\end{Bmatrix}=\begin{Bmatrix}0\\ 0\\ 0\\ \mathcal{F}_1^4\\ \mathcal{F}_2^4\end{Bmatrix} \tag{87}$$

We now add ALL of these elements to get our global system,

$$\frac{AE}{L}\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix} = \begin{Bmatrix} \mathcal{F}_1^1 \\ \mathcal{F}_2^1 + \mathcal{F}_1^2 \\ \mathcal{F}_2^2 + \mathcal{F}_1^3 \\ \mathcal{F}_2^3 + \mathcal{F}_1^4 \\ \mathcal{F}_2^4 \end{Bmatrix} \tag{88}$$

Look Close at the right hand side. The right hand side consist of two terms, equation 68 and 69, repeated here with alterations.

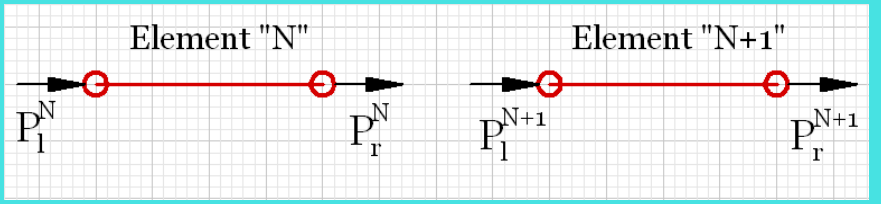
$$\mathcal{F}_1^e = \left[\int_0^{L^e} N_1^e(x)\, S(x)\, dx - P_l^e\right] = \mathcal{G}_1^e - P_l^e \tag{89}$$

$$\mathcal{F}_2^e = \left[\int_0^{L^e} N_2^e(x)\, S(x)\, dx + P_r^e\right] = \mathcal{G}_2^e + P_r^e \tag{90}$$

With this splitting our equation becomes,

$$\frac{AE}{L}\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix} = \begin{Bmatrix} \mathcal{G}_1^1 \\ \mathcal{G}_2^1 + \mathcal{G}_1^2 \\ \mathcal{G}_2^2 + \mathcal{G}_1^3 \\ \mathcal{G}_2^3 + \mathcal{G}_1^4 \\ \mathcal{G}_2^4 \end{Bmatrix} + \begin{Bmatrix} -P_l^1 \\ P_r^1 - P_l^2 \\ P_r^2 - P_l^3 \\ P_r^3 - P_l^4 \\ P_r^4 \end{Bmatrix} \tag{91}$$

If we look at the figure below we see these terms relating to the applied force cancel out,



Our final matrix becomes,

$$\frac{AE}{L}\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix} = \begin{Bmatrix} \mathcal{G}_1^1 \\ \mathcal{G}_2^1 + \mathcal{G}_1^2 \\ \mathcal{G}_2^2 + \mathcal{G}_1^3 \\ \mathcal{G}_2^3 + \mathcal{G}_1^4 \\ \mathcal{G}_2^4 \end{Bmatrix} + \begin{Bmatrix} -P_l^1 \\ 0 \\ 0 \\ 0 \\ P_r^4 \end{Bmatrix} \tag{92}$$

Our matrix is actually singular at this point and can not be inverted. However, our boundary conditions come in to play. First, we will solve our model problem, repeated here.

$$AE\frac{d^2U}{dx^2} + S(x) = 0 \tag{93}$$

Where the term $S(x)$, is an arbitrary function of x. We shall also take our boundary conditions as,

$$U(x=0) = U(x=1) = 0 \tag{94}$$

For our problem we will take our source function $S(x)$ as,

$$S(x) = -25\sin(4\pi x) - 33\sin(3\pi x) \tag{95}$$

We set $AE = 1$.

For our 4 element finite element solution, matrix above, how do we set our boundary conditions?

$$U(x=0) = 0 \Longrightarrow U_1 = 0 \tag{96}$$

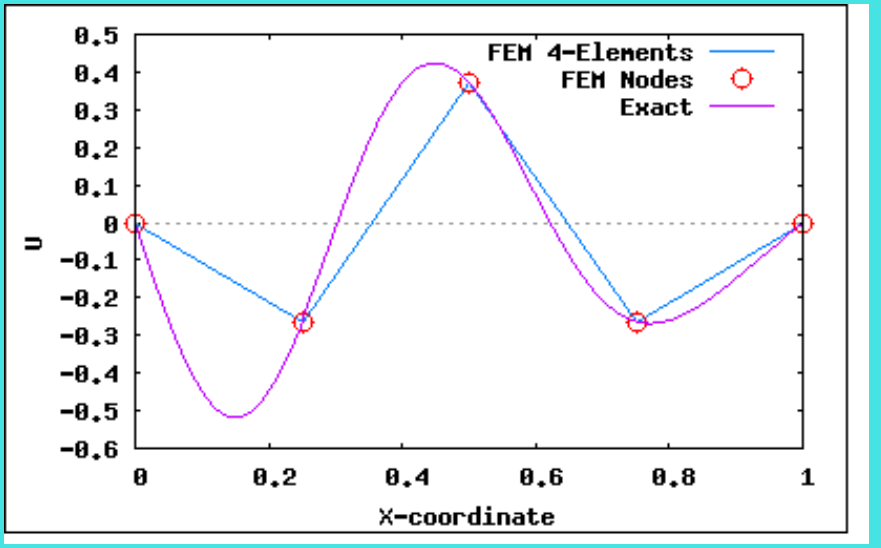
$$U(x=1) = 0 \Longrightarrow U_5 = 0 \tag{97}$$

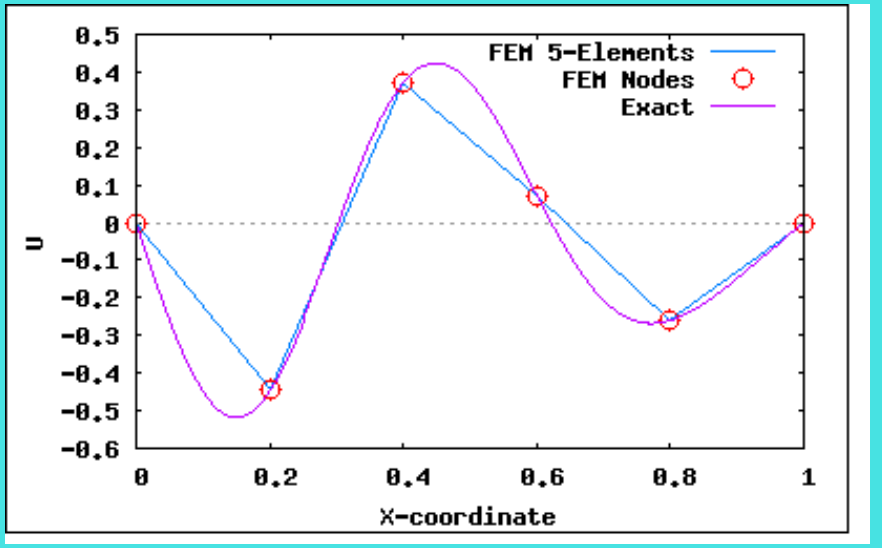
Doing so results in the following matrix equation,

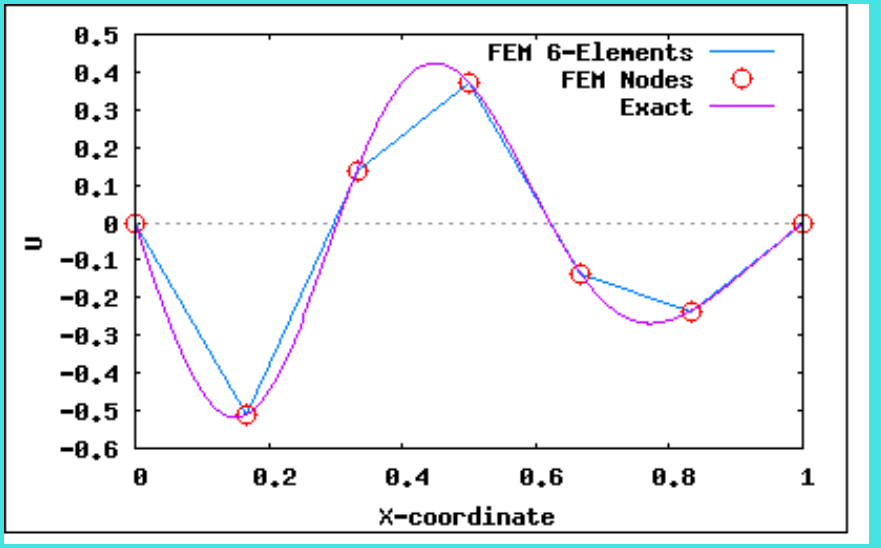
$$\frac{AE}{L}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \mathcal{G}_2^1 + \mathcal{G}_1^2 \\ \mathcal{G}_2^2 + \mathcal{G}_1^3 \\ \mathcal{G}_2^3 + \mathcal{G}_1^4 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{98}$$

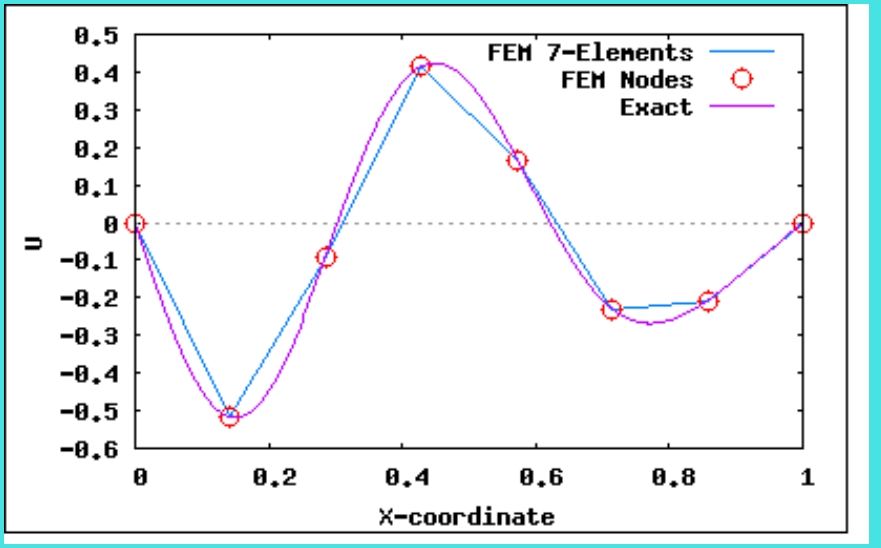
We completely replace the first and last equation with our Boundary Condition.

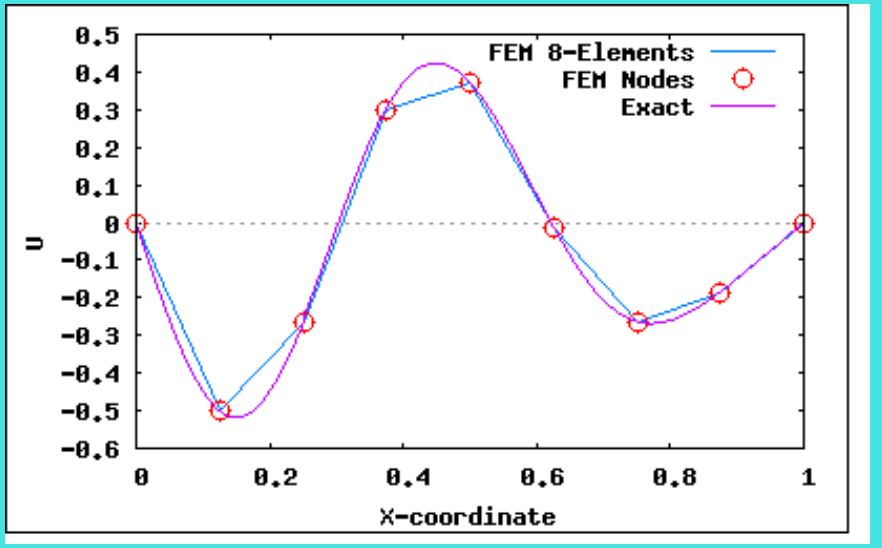
We now run off to Maxima to execute a solution.

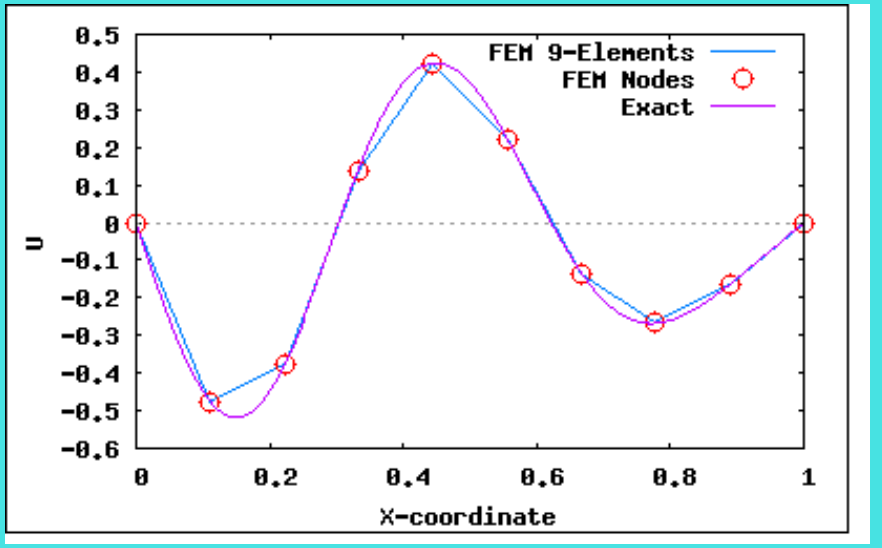

FEM 4-Elements
FEM Nodes
Exact
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
U
0
0.2
0.4
0.6
0.8
1
X-coordinate

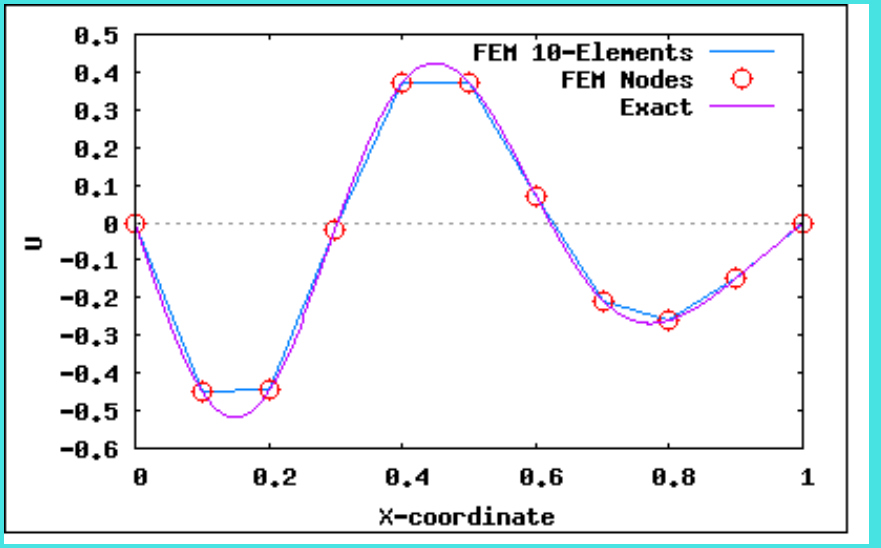

FEM 5-Elements
FEM Nodes
Exact
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
U
0
0.2
0.4
0.6
0.8
1
X-coordinate

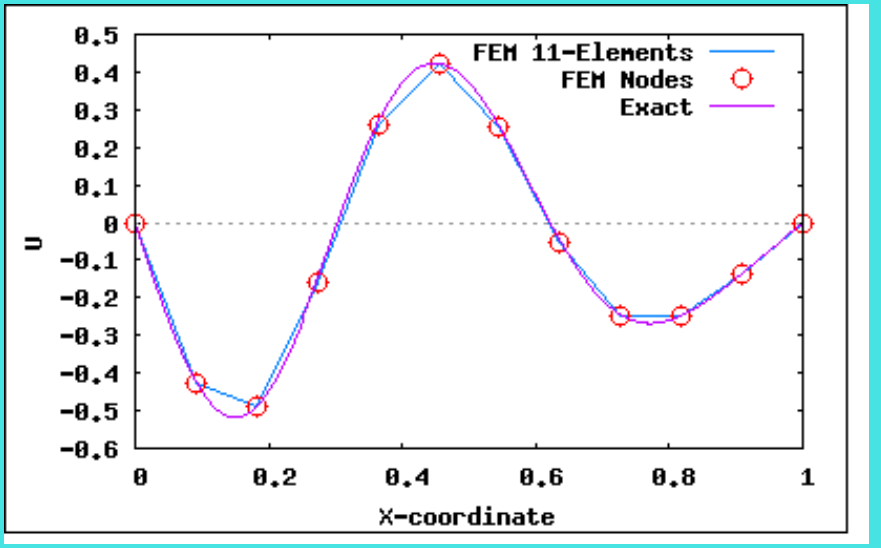

FEM 6-Elements
FEM Nodes
Exact
U
X-coordinate
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
0
0.2
0.4
0.6
0.8
1

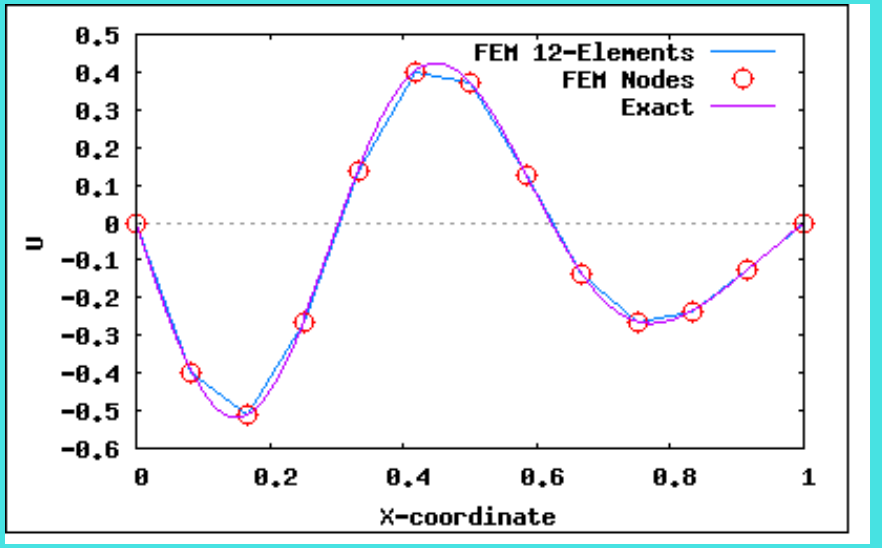

FEM 7-Elements
FEM Nodes
Exact
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
U
0
0.2
0.4
0.6
0.8
1
X-coordinate


FEM 8-Elements
FEM Nodes
Exact
U
X-coordinate
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
0
0.2
0.4
0.6
0.8
1


FEM 9-Elements
FEM Nodes
Exact
U
X-coordinate
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
0
0.2
0.4
0.6
0.8
1

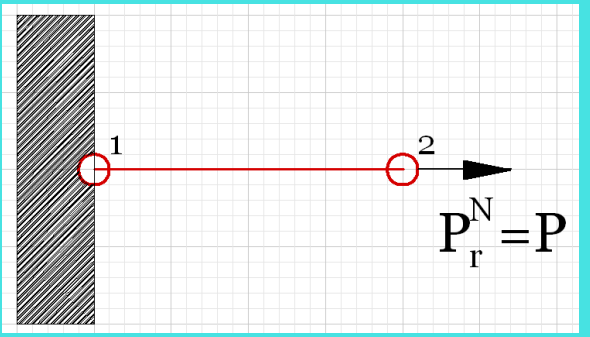

FEM 10-Elements
FEM Nodes
Exact
U
X-coordinate
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
0
0.2
0.4
0.6
0.8
1


FEM 11-Elements
FEM Nodes
Exact
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
U
0
0.2
0.4
0.6
0.8
1
X-coordinate


FEM 12-Elements
FEM Nodes
Exact
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
U
0
0.2
0.4
0.6
0.8
1
X-coordinate

2.3. Comment

It may be of some interest to investigate a one element solution as shown below.



With the boundary condition on the left begin a zero displacement our system of equations, only one element matrix, becomes,

$$\frac{AE}{L}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ P \end{Bmatrix} \tag{99}$$

Note, there is no source term. Simplifying this we get two equations,

$$U_1 = 0 \tag{100}$$

$$\frac{AE}{L}U_2 = P \tag{101}$$

OR

$$U_2 = \frac{PL}{AE} \tag{102}$$

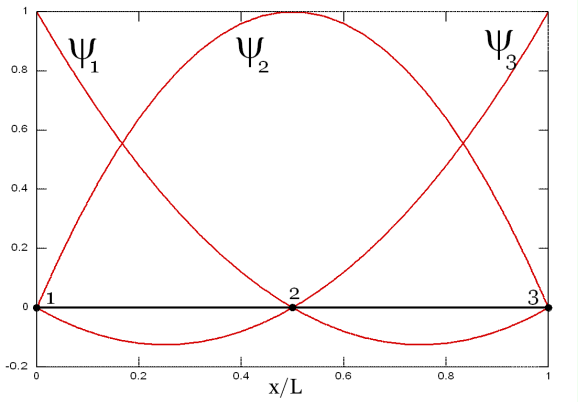
The standard solution in Mechanics of Materials textbooks.

3. Quadratic Elements

We will add a quick section on quadratic element along with a quadratic varying applied load. The purpose is to bring our bar analysis in sync, so to speak, with our beam analysis. In the next course we will see the use of elements with 3 nodes for beams. Our equation of interest is,

$$\frac{d}{dx}\left[EA\frac{dU}{dx}\right] = a_0 + a_1 x \tag{103}$$

If we represent our elemental functions through quadratic functions we need 3 nodes in each element.



These functions are,

$$\begin{aligned}
\psi_1^e(x) &= \left(1-\frac{x}{L}\right)\left(1-\frac{2x}{L}\right)\\
\psi_2^e(x) &= \frac{4x}{L}\left(1-\frac{x}{L}\right)\\
\psi_3^e(x) &= -\frac{x}{L}\left(1-\frac{2x}{L}\right)
\end{aligned} \tag{104}$$

Using these functions we can determine our stiffness matrix and right hand side as,

$$K_{ij}^e = \int_0^L \frac{d\psi_i}{dx} EA \frac{d\psi_j}{dx} dx \tag{105}$$

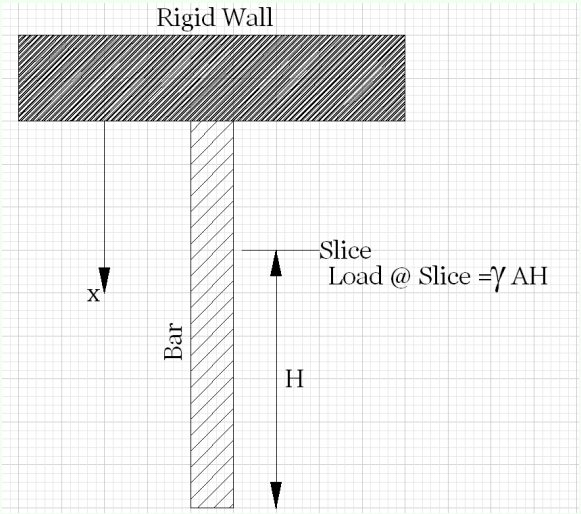
$$f_i^e = \int_0^L \frac{d\psi_i}{dx}\,(a_0 + a_1 x)\,dx \tag{106}$$

Carrying out this integration we obtain

$$[K^e] = \frac{EA}{3L}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} \tag{107}$$

$$\{f^e\} = \frac{a_0 L}{6}\begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix} + \frac{q_1 L^2}{6}\begin{Bmatrix} 0 \\ 2 \\ 1 \end{Bmatrix} \tag{108}$$

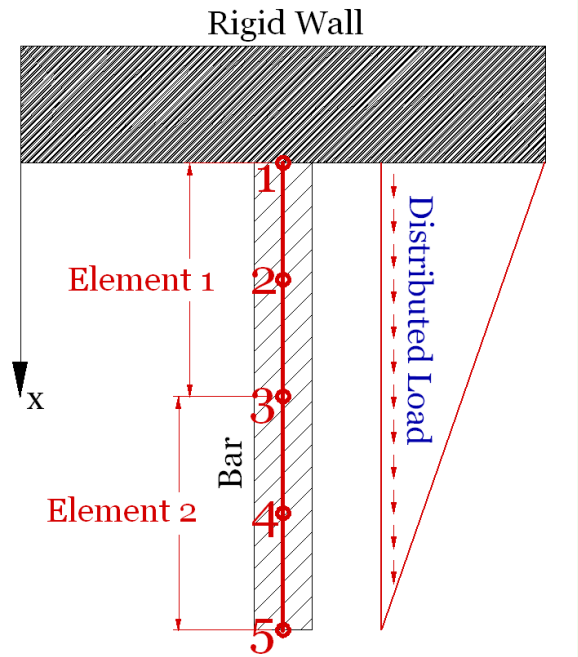
3.1. Bar deflecting under its own weight



At the end of the bar $x = L$ there is no load. However, as we move up the bar, $x < L$ the portion below the bar is pulling down, in the positive x direction. We have,

$$\frac{d}{dx}\left[EA\frac{dU}{dx}\right] = \beta_b - \frac{\beta_b}{L}x \tag{109}$$

Where β_b is the distributed load on the bar. And, $\left(\beta_b - \frac{\beta_b}{L}x\right)$ is constructed such that this term is zero when $x = L$. Note: since the load is a triangularly distributed load the total load is $W_b = \beta L/2$. This means for our Bar to weight W_b, our β term must be, $\beta = 2W_b/L$. If we use two elements to solve our problem,



Our Boundary conditions are,

- $U_1 = 0$
- $P_5 = 0$

Lets make our case complete by using,

- $L = 10$

- $EA = 1000$
- $\gamma AL = 100$ pounds. I.E. the total weight of the bar is 100 pounds.

We now write each of our element equations.

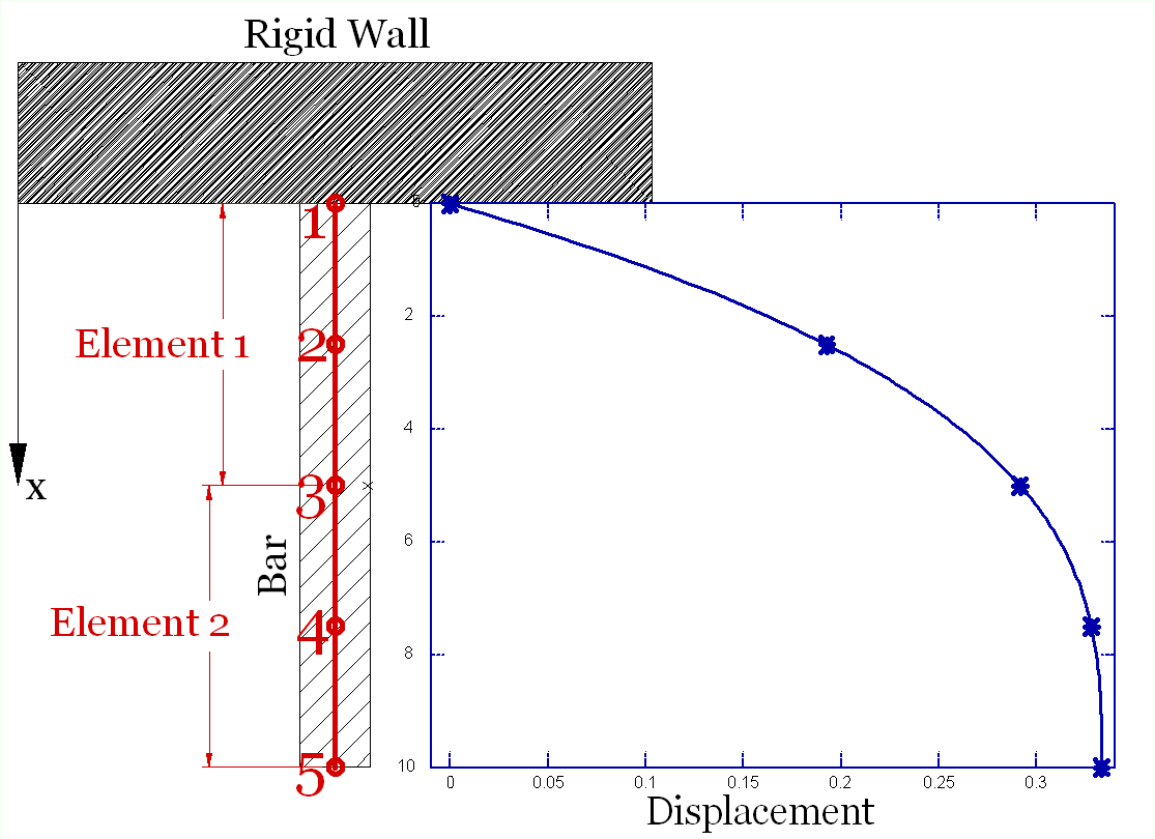
Element 1

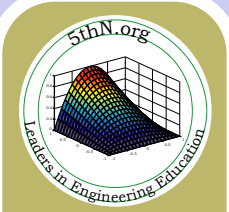
$$\frac{EA}{3L}\begin{bmatrix} 7 & -8 & 1 & 0 & 0 \\ -8 & 16 & -8 & 0 & 0 \\ 1 & -8 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix} = \frac{[20]\,L}{6} \begin{Bmatrix} 1 \\ 4 \\ 1 \\ 0 \\ 0 \end{Bmatrix} + \frac{[-2]\,L^2}{6} \begin{Bmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{Bmatrix} \tag{110}$$

Element 2

$$\frac{EA}{3L}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 7 & -8 & 1 \\ 0 & 0 & -8 & 16 & -8 \\ 0 & 0 & 1 & -8 & 7 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix} = \frac{[10]\,L}{6} \begin{Bmatrix} 0 \\ 0 \\ 1 \\ 4 \\ 1 \end{Bmatrix} + \frac{[-2]\,L^2}{6} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 1 \end{Bmatrix} \tag{111}$$

A plot of the displacement is shown on the next slide. As we can see, the slop of the deflection is zero at the end of the bar, signifying the absence of an applied load.


Rigid Wall
Element 1
Element 2
Bar
x
1
2
3
4
5
0
2
4
6
8
10
0
0.05
0.1
0.15
0.2
0.25
0.3
Displacement



THE END

Mathematics of FEM

- Next Course Deals with Beams!
- Plane Frame Course Deals With Bar-Beam Combo.
- Thank You for Training With US

http://5thn.org/