

Finite Elements of Bars & Beams

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Abstract

In this course we will use Maxima to develop the Finite Element Method for bars and beams. We will,

- Reviewing the development of the FEM equations for our bar equation.
- Cover the assembly process of our bar based element equations.
- Investigate the equations for beam deflection.
- Case our beam equation in a finite element weighted residual statement.
- Convert our beam equation using integration by parts, twice.
- Investigate the beam boundary conditions generated from our integration by parts.
- Review Beam deflections, shear and moment diagrams from mechanics of materials.
- Develop the finite element equations for beam equations.
- Compare Finite Element TO mechanics of materials solutions.
- Warning, deficiencies of the finite element method.
- Development of a hierarchial element function to correct the deficiencies of the finite element method.

http://5thn.org/

1. Model Problem

In the previous PDH lecture we investigated the bar equation,

$$\frac{d}{dx}\left[AE\frac{dU}{dx}\right]+S\left(x\right)=0 \tag{1}$$

Where the term $S\left(x\right)$, is an arbitrary loading function of x. Our boundary conditions are given in two types. Either a fixed displacement or an end load. Examples of which could be,

$$\begin{aligned} U\left(x=0\right)&=0\\ \left[AU\frac{dU}{dx}\right]_{x=L_{bar}}&=P \end{aligned} \tag{2}$$

For problems we shall consider from here, the source function $S\left(x\right)$ is zero. Hence our equations become,

$$\frac{d}{dx}\left[AE\frac{dU}{dx}\right]=0 \tag{3}$$

From our last PDH lecture we obtained a weighted residual statement of,

$$\sum_{e=1}^{N_{elem}}\left[\int_0^{L^e}N_i^e\left(x\right)\frac{d}{dx}\left[AE\frac{dU\left(x\right)}{dx}\right]dx\right]=0 \tag{4}$$

Where,

$$U\left(x\right)=\sum_{i=1}^{2}N_i^e\left(x\right)U_i^e=N_1^e\left(x\right)U_1^e+N_2^e\left(x\right)U_2^e \tag{5}$$

And

$$N_1^e\left(x\right)=\left(1-\frac{x}{L^e}\right) \tag{6}$$

$$N_2^e\left(x\right)=\frac{x}{L^e} \tag{7}$$

Performing and integration by parts on our equation we obtain 2 equations,

$$\mathcal{E}_1^e = -P_l - \sum_{j=1}^{2}\left[\int_0^{L^e} \frac{dN_1^e(x)}{dx} AE \frac{dN_j^e(x)}{dx} dx\right] U_j^e \tag{8}$$

$$\mathcal{E}_2^e = P_r - \sum_{j=1}^{2}\left[\int_0^{L^e} \frac{dN_2^e(x)}{dx} AE \frac{dN_j^e(x)}{dx} dx\right] U_j^e \tag{9}$$

Hence we obtain a matrix equation,

$$\frac{AE}{L^e}\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}\begin{Bmatrix} U_1^e \\ U_2^e \end{Bmatrix} = \begin{Bmatrix} -P_l \\ P_r \end{Bmatrix} \tag{10}$$

Our derivatives are,

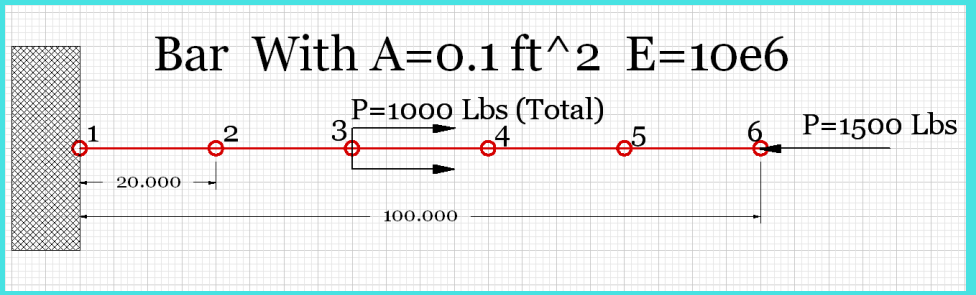
$$\frac{dN_1^e}{dx} = -\frac{1}{L^e} \tag{11}$$

Hence our equations become,

$$\frac{dN_2^e}{dx} = \frac{1}{L^e} \tag{12}$$

$$\frac{AE}{L^e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} U_1^e \\ U_2^e \end{Bmatrix} = \begin{Bmatrix} -P_l \\ P_r \end{Bmatrix} \tag{13}$$

If we take a simple example as shown in the figure below,



In this example each the 5 elements are of length 20 ft. The area, A, is 0.1 ft^2 and the modulus is that of aluminum, $E = 10e6\ \frac{lbs}{in^2}$. As a result each element matrix equation is,

$$\frac{0.1 \times 10e6 \times 144}{20} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} U_1^e \\ U_2^e \end{Bmatrix} = 7.2e6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} U_1^e \\ U_2^e \end{Bmatrix} = \begin{Bmatrix} -P_l \\ P_r \end{Bmatrix} \tag{14}$$

Element 1,

$$7.2e6 \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} = \begin{Bmatrix} -P_l^1 \\ P_r^1 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{15}$$

Element 2,

$$7.2e6 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -P_l^2 \\ P_r^2 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{16}$$

Element 3,

$$7.2e6 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -P_l^3 \\ P_r^3 \\ 0 \\ 0 \end{Bmatrix} \tag{17}$$

Element 4,

$$7.2e6 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -P_l^4 \\ P_r^4 \\ 0 \end{Bmatrix} \tag{18}$$

Element 5,

$$7.2e6\begin{bmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&1&-1\\0&0&0&0&-1&1\end{bmatrix}\begin{Bmatrix}U_1\\U_2\\U_3\\U_4\\U_5\\U_6\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\\0\\-P_l^5\\P_r^5\end{Bmatrix} \tag{19}$$

Summing all of our element equations together we get the following equation,

$$7.2e6\begin{bmatrix}1&-1&0&0&0&0\\-1&2&-1&0&0&0\\0&-1&2&-1&0&0\\0&0&-1&2&-1&0\\0&0&0&-1&2&-1\\0&0&0&0&-1&1\end{bmatrix}\begin{Bmatrix}U_1\\U_2\\U_3\\U_4\\U_5\\U_6\end{Bmatrix}=\begin{Bmatrix}-P_l^1\\P_r^1-P_l^2\\P_r^2-P_l^3\\P_r^3-P_l^4\\P_r^4-P_l^5\\P_r^5\end{Bmatrix} \tag{20}$$

Noticing our applied loads at node 3 and 6 our right hand side becomes,

$$\begin{Bmatrix}-P_l^1\\P_r^1-P_l^2\\P_r^2-P_l^3\\P_r^3-P_l^4\\P_r^4-P_l^5\\P_r^5\end{Bmatrix}=\begin{Bmatrix}-P_l^1\\0\\1000\\0\\0\\-1500\end{Bmatrix} \tag{21}$$

The load on node remains because there is a load there from the wall. Even though we know it will be 500 pounds to the left, the finite element should tell us its value! Our matrix equation becomes,

$$7.2e6\begin{bmatrix}1&-1&0&0&0&0\\-1&2&-1&0&0&0\\0&-1&2&-1&0&0\\0&0&-1&2&-1&0\\0&0&0&-1&2&-1\\0&0&0&0&-1&1\end{bmatrix}\begin{Bmatrix}U_1\\U_2\\U_3\\U_4\\U_5\\U_6\end{Bmatrix}=\begin{Bmatrix}-P_l^1\\0\\1000\\0\\0\\-1500\end{Bmatrix} \tag{22}$$

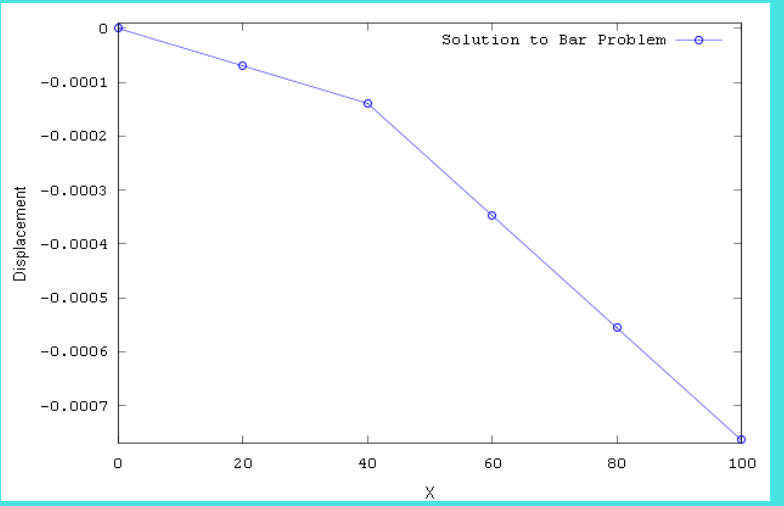
We must specify our boundary condition, $U_1 = 0$. This is done by removing our first equation from the matrix to give,

$$7.2e6 \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 1000 \\ 0 \\ 0 \\ -1500 \end{Bmatrix} \tag{23}$$

Solving this system we get the displacements.

$$\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} = \begin{Bmatrix} +1.6263e-019 \\ -6.9444e-005 \\ -1.3889e-004 \\ -3.4722e-004 \\ -5.5556e-004 \\ -7.6389e-004 \end{Bmatrix} \tag{24}$$

A plot of this displacement field is shown below,

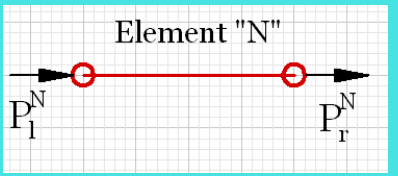


1.1. Post Processing

If we take our solution and place the displacements in our original matrix equation, 22, or Global loads become,

$$7.2e6\begin{bmatrix}1&-1&0&0&0&0\\-1&2&-1&0&0&0\\0&-1&2&-1&0&0\\0&0&-1&2&-1&0\\0&0&0&-1&2&-1\\0&0&0&0&-1&1\end{bmatrix}\begin{Bmatrix}+1.6263e-019\\-6.9444e-005\\-1.3889e-004\\-3.4722e-004\\-5.5556e-004\\-7.6389e-004\end{Bmatrix}=\begin{Bmatrix}500\\0\\1000\\0\\0\\-1500\end{Bmatrix} \tag{25}$$

As expected, our load on the left end where the bar connects to the wall is 500 pounds. The right hand side of this equation is the loads applied at each global node. Now we begin to investigate element by element. If we look at any element "N" in our grid we see the following,



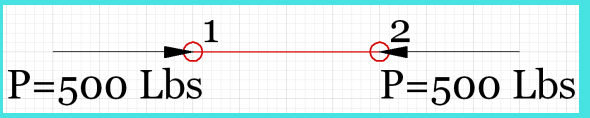
If we go back to our element equations and place our displacements in our equations we obtain, for element 1,

$$7.2e6\begin{bmatrix}1&-1\\-1&1\end{bmatrix}\begin{Bmatrix}-6.9444e-005\\-1.3889e-004\end{Bmatrix}=\begin{Bmatrix}500\\-500\end{Bmatrix} \tag{26}$$

For element 2,

$$7.2e6\begin{bmatrix}1&-1\\-1&1\end{bmatrix}\begin{Bmatrix}+1.6263e-019\\-6.9444e-005\end{Bmatrix}=\begin{Bmatrix}500\\-500\end{Bmatrix} \tag{27}$$

For elements 1 and 2 the free body diagram for both elements look like,



Notice the direction of the force on each end. One the left end the 500 pound load is in the direct of the x coordinate, hence its value is positive. On the right end the 500 pound load is to the left, in the negative x direction.

For element 3, we have

$$7.2e6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} -1.3889e-004 \\ -3.4722e-004 \end{Bmatrix} = \begin{Bmatrix} 1500 \\ -1500 \end{Bmatrix} \tag{28}$$

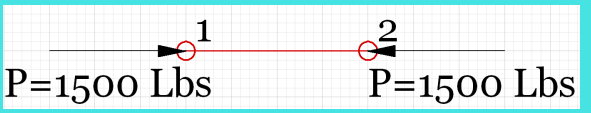
For element 4, we have

$$7.2e6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} -3.4722e-004 \\ -5.5556e-004 \end{Bmatrix} = \begin{Bmatrix} 1500 \\ -1500 \end{Bmatrix} \tag{29}$$

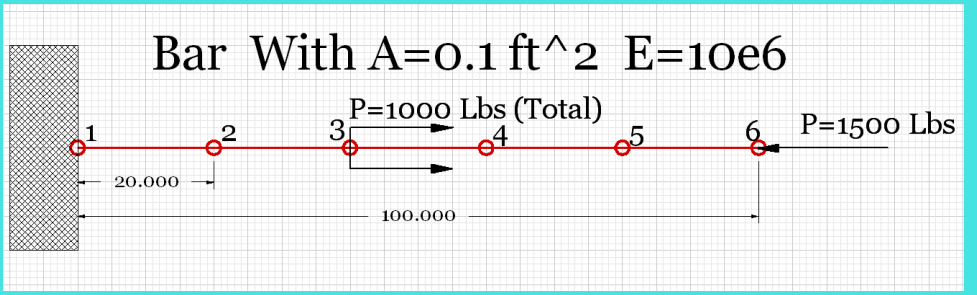
For element 5, we have

$$7.2e6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} -5.5556e-004 \\ -7.6389e-004 \end{Bmatrix} = \begin{Bmatrix} 1500 \\ -1500 \end{Bmatrix} \tag{30}$$

For elements 3, 4 and 5 the free body diagram for both elements look like,



NOTE: In all both these figures, nodes 1 and 2 are local node numbers.
Global,



FBD $\Longrightarrow$ Slice by section...

2. BEAMS

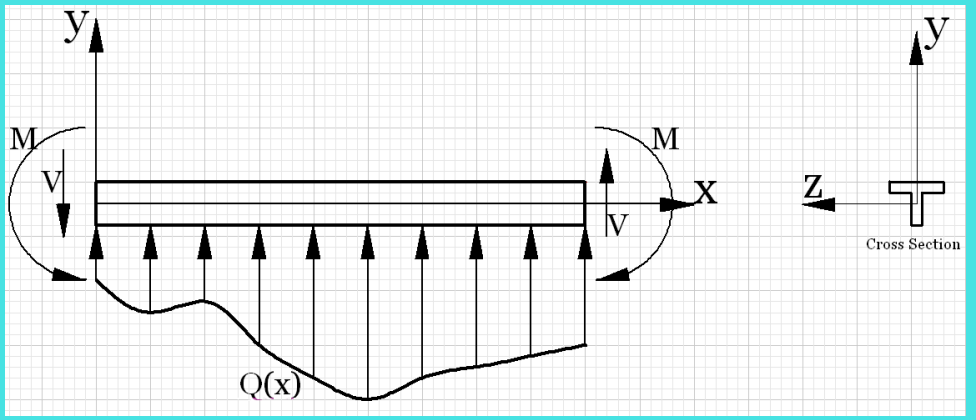
Euler-Bernoulli beam theory may be described using the following differential equation.

$$\frac{d^2}{dx^2}\left[EI\frac{d^2W(x)}{dx^2}\right] = Q(x) \tag{31}$$

Where

- $W(x)$ is the displacement of the beam as a function of x.
- E is the modulus of elasticity.
- I is the second moment of the area of the cross section of the beam.
- $Q(x)$ is the distributed load on the beam.

Where we are using the following coordinate systems,



Note: From Mechanics of Materials we have,

$$M = -EI\frac{d^2W}{dx^2} \quad : \quad V = \frac{dM}{dx} = -\frac{d}{dx}\left[EI\frac{d^2W}{dx^2}\right] \quad : \quad -\frac{dV}{dx} = Q \tag{32}$$

If we follow the standard weighted residual method we should form the following statement,

$$\sum_{e=1}^{N_{elem}} \left[\int_{x_l}^{x_r} H_i^e(x) \left\{ \frac{d^2}{dx^2} \left[EI \frac{d^2 W(x)}{dx^2} \right] - Q(x) \right\} dx \right] = 0 \tag{33}$$

Where $H_i^e(x)$ is the functions we will be using in our approximation, (Comment Φ_i^e.)

$$W(x) = \sum_{i=1}^{N} H_i^e(x) \Phi_i^e \tag{34}$$

We shall focus on one term and use integration by parts to obtain,

$$\begin{aligned} \int_{x_l}^{x_r} H_i^e(x) \frac{d^2}{dx^2} \left[EI \frac{d^2 W(x)}{dx^2} \right] dx = & - \int_{x_l}^{x_r} \frac{dH_i^e(x)}{dx} \frac{d}{dx} \left[EI \frac{d^2 W(x)}{dx^2} \right] dx \\ & - \left[H_i^e(x) \underbrace{\left(-\frac{d}{dx} \left[EI \frac{d^2 W(x)}{dx^2} \right] \right)}_{V(x)} \right]_{x_l}^{x_r} \end{aligned} \tag{35}$$

Where is $V(x)$ is the standard definition of the shear in a beam.

$$\begin{aligned} \int_{x_l}^{x_r} H_i^e(x) \frac{d^2}{dx^2} \left[EI \frac{d^2 W(x)}{dx^2} \right] dx = & - \int_{x_l}^{x_r} \frac{dH_i^e(x)}{dx} \frac{d}{dx} \left[EI \frac{d^2 W(x)}{dx^2} \right] dx \\ & - \left[H_i^e(x) V(x) \right]_{x_l}^{x_r} \end{aligned} \tag{36}$$

Integrating by parts once more we get,

$$\begin{aligned} \int_{x_l}^{x_r} H_i^e(x) \frac{d^2}{dx^2} \left[EI \frac{d^2 W(x)}{dx^2} \right] dx = & \int_{x_l}^{x_r} \frac{d^2 H_i^e(x)}{dx^2} EI \frac{d^2 W(x)}{dx^2} dx \\ & - \left[H_i^e(x) V(x) \right]_{x_l}^{x_r} + \left[\frac{dH_i^e(x)}{dx} \underbrace{\left(-EI \frac{d^2 W(x)}{dx^2} \right)}_{M(x)} \right]_{x_l}^{x_r} \end{aligned} \tag{37}$$

Where $M(x)$ is the standard definite of the moment in beam diagrams.

$$\int_{x_l}^{x_r} H_i^e(x) \frac{d^2}{dx^2}\left[EI\frac{d^2W(x)}{dx^2}\right]dx = \int_{x_l}^{x_r} \frac{d^2H_i^e(x)}{dx^2} EI \frac{d^2W(x)}{dx^2}dx - \left[H_i^e(x)V(x)\right]_{x_l}^{x_r} + \left[\frac{dH_i^e(x)}{dx}M(x)\right]_{x_l}^{x_r} \tag{38}$$

With these integration by parts applied our element equation becomes,

$$\int_{x_l}^{x_r} \frac{d^2H_i^e(x)}{dx^2} EI \frac{d^2W(x)}{dx^2}dx = \int_{x_l}^{x_r} H_i^e(x)Q(x)\,dx - \left[\frac{dH_i^e(x)}{dx}M(x)\right]_{x_l}^{x_r} + \left[H_i^e(x)V(x)\right]_{x_l}^{x_r} \tag{39}$$

We now pause to investigate our basis functions. Without much proof, saved for later, we will introduce the Hermite cubic functions as our functions to be used in each element. Remember our beam displacement is represented as,

$$W(x) = \sum_{i=1}^{4} H_i^e(x)\Phi_i^e \tag{40}$$

Where our Hermite cubic functions are,

$$\begin{aligned} H_1^e(\tilde{x}) &= 1 - \frac{3\tilde{x}^2}{L^2} + \frac{2\tilde{x}^3}{L^3} \\ H_2^e(\tilde{x}) &= \tilde{x} - \frac{2\tilde{x}^2}{L} + \frac{\tilde{x}^3}{L^2} \\ H_3^e(\tilde{x}) &= \frac{3\tilde{x}^2}{L^2} - \frac{2\tilde{x}^3}{L^3} \\ H_4^e(\tilde{x}) &= \frac{\tilde{x}^3}{L^2} - \frac{\tilde{x}^2}{L} \end{aligned} \tag{41}$$

We list the derivatives of these functions as,

$$\begin{aligned}\frac{dH_1^e}{d\tilde{x}} &= \frac{6\tilde{x}^2}{L^3} - \frac{6\tilde{x}}{L^2} \\ \frac{dH_2^e}{d\tilde{x}} &= 1 - \frac{4\tilde{x}}{L} + \frac{3\tilde{x}^2}{L^2} \\ \frac{dH_3^e}{d\tilde{x}} &= \frac{6\tilde{x}}{L^2} - \frac{6\tilde{x}^2}{L^3} \\ \frac{dH_4^e}{d\tilde{x}} &= \frac{3\tilde{x}^2}{L^2} - \frac{2\tilde{x}}{L}\end{aligned} \tag{42}$$

One very interesting and very important property follows,

$$\begin{aligned}H_1^e\left(\tilde{x}=0\right) &= 1 \\ H_2^e\left(\tilde{x}=0\right) &= 0 \\ H_3^e\left(\tilde{x}=0\right) &= 0 \\ H_4^e\left(\tilde{x}=0\right) &= 0\end{aligned} \tag{43}$$

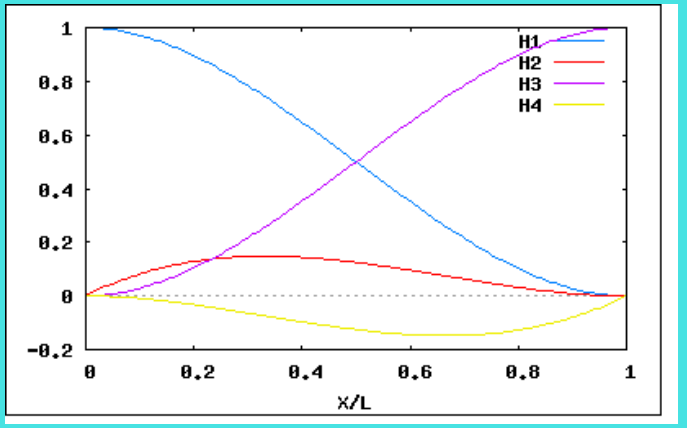
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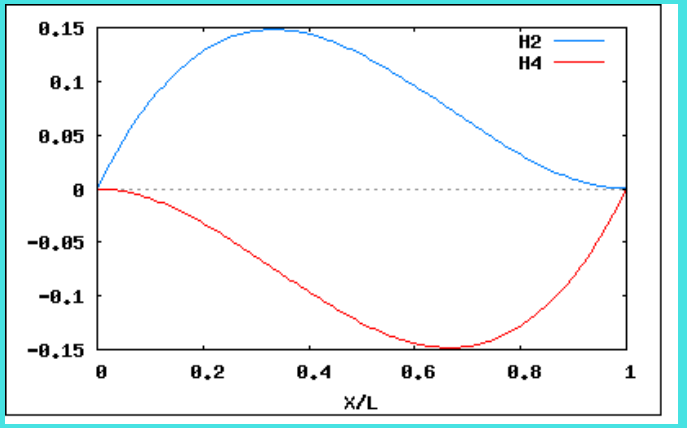
$$\begin{aligned}H_1^e\left(\tilde{x}=L\right) &= 0 \\ H_2^e\left(\tilde{x}=L\right) &= 0 \\ H_3^e\left(\tilde{x}=L\right) &= 1 \\ H_4^e\left(\tilde{x}=L\right) &= 0\end{aligned} \tag{44}$$

$$\begin{aligned}\left[\frac{dH_1^e}{d\tilde{x}}\right]_{\tilde{x}=0} &= 0 \\ \left[\frac{dH_2^e}{d\tilde{x}}\right]_{\tilde{x}=0} &= 1 \\ \left[\frac{dH_3^e}{d\tilde{x}}\right]_{\tilde{x}=0} &= 0 \\ \left[\frac{dH_4^e}{d\tilde{x}}\right]_{\tilde{x}=0} &= 0\end{aligned} \tag{45}$$

$$\left[\frac{dH_1^e}{d\tilde{x}}\right]_{\tilde{x}=L} = 0$$
$$\left[\frac{dH_2^e}{d\tilde{x}}\right]_{\tilde{x}=L} = 0$$
$$\left[\frac{dH_3^e}{d\tilde{x}}\right]_{\tilde{x}=L} = 0$$
$$\left[\frac{dH_4^e}{d\tilde{x}}\right]_{\tilde{x}=L} = 1 \qquad (46)$$

Much can be observed about these functions from,

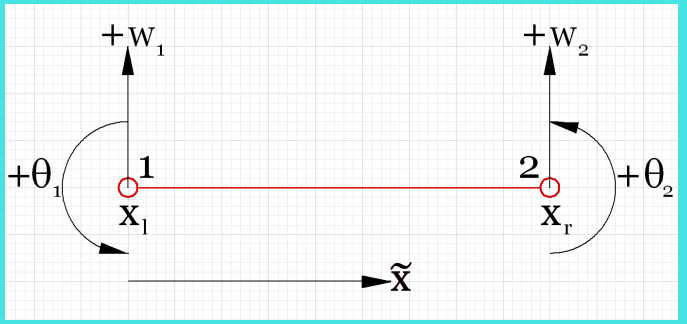




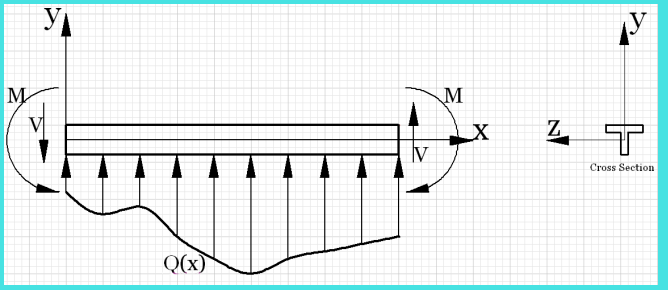
As a result of these functions our approximation may be written as,

$$W(x) = H_1^e(x) W_1^e + H_2^e(x) \theta_1^e + H_3^e(x) W_2^e + H_4^e(x) \theta_2^e \tag{47}$$

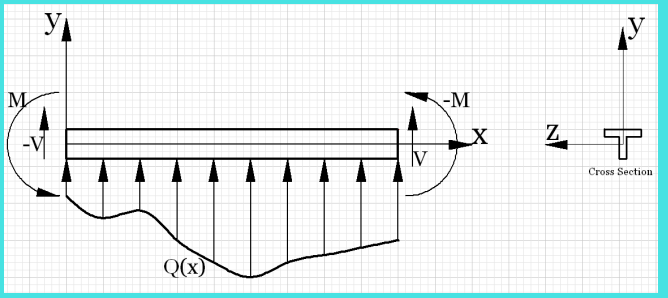
Where the displacement and rotations are shown in the figure below,



As we see by comparing this plot to our coordinate,



we could redo this plot as,



To make the shear and moments align with the definitions of displacement and rotations. We shall experiment with this information using several cases to solidify understanding with respect to applied loads and resulting deflections. In addition these experiments will help understand postprocessing results.

Also a comment should be made about the differences between x, $\tilde{x}$, $_l$ and x_r. There is a relationship between all of these terms,

$$x = x_l + \tilde{x} \tag{48}$$

Where x is the actual location in any element. x_l is the coordinate of the left end of the element. $\tilde{x}$ is the coordinate within any element. This value is in the range of 0 to L, the length of the beam. Hence the following relationships hold,

$$\begin{aligned} x = x_l &\Longrightarrow \tilde{x} = 0 \\ x = x_r &\Longrightarrow \tilde{x} = L \end{aligned} \tag{49}$$

Repeating equation 39 here we notice several things,

$$\int_{x_l}^{x_r} \frac{d^2 H_i^e(x)}{dx^2} EI \frac{d^2 W(x)}{dx^2} dx = \int_{x_l}^{x_r} H_i^e(x) Q(x) dx - \left[\frac{dH_i^e(x)}{dx} M(x) \right]_{x_l}^{x_r} + \left[H_i^e(x) V(x) \right]_{x_l}^{x_r} \quad (50)$$

Using the exchange of variables,

$$\begin{aligned} x = x_l &\Longrightarrow \tilde{x} = 0 \\ x = x_r &\Longrightarrow \tilde{x} = L \end{aligned} \quad (51)$$

Our equation becomes

$$\int_0^{L^e} \frac{d^2 H_i^e(x)}{dx^2} EI \frac{d^2 W(x)}{dx^2} dx = \int_0^{L^e} H_i^e(x) Q(x) dx - \left[\frac{dH_i^e(x)}{dx} M(x) \right]_0^{L^e} + \left[H_i^e(x) V(x) \right]_0^{L^e} \quad (52)$$

First, there are actually 4 equations $i = 1, 2, 3, 4$, where we save space by $H_i^e(x) \Longrightarrow H_i^e$ under the integrals to obtain.

$$\int_0^{L^e} \frac{d^2 H_1^e}{dx^2} EI \frac{d^2 W}{dx^2} dx = \int_0^{L^e} H_1^e Q(x) dx - \left[\frac{dH_1^e(x)}{dx} M(x) \right]_0^{L^e} + \left[H_1^e(x) V(x) \right]_0^{L^e} \quad (53)$$

OR, using the relationships from our functions 3 slides back we have,

$$\int_0^{L^e} \frac{d^2 H_1^e}{dx^2} EI \frac{d^2 W}{dx^2} dx = \int_0^{L^e} H_1^e Q(x) dx - \left[\underbrace{\frac{dH_1^e}{dx}}_{=0} M \right]_{L^e} + \left[\underbrace{\frac{dH_1^e}{dx}}_{=0} M \right]_0 + \left[\underbrace{H_1^e}_{=0} V \right]_{L^e} - \left[\underbrace{H_1^e}_{=1} V \right]_0 \quad (54)$$

Or

$$\boxed{\int_0^{L^e} \frac{d^2 H_1^e}{dx^2} EI \frac{d^2 W}{dx^2} dx = \int_0^{L^e} H_1^e Q(x) dx - V_l^e} \quad (55)$$

Where V_l^e is the shear on the left end of the element.

Our second equation may be written as,

$$\int_0^{L^e} \frac{d^2 H_2^e}{dx^2} EI \frac{d^2 W}{dx^2} dx = \int_0^{L^e} H_2^e Q(x) dx - \left[\underbrace{\frac{dH_2^e}{dx}}_{=0} M \right]_{L^e} + \left[\underbrace{\frac{dH_2^e}{dx}}_{=1} M \right]_0 + \left[\underbrace{H_2^e}_{=0} V \right]_{L^e} - \left[\underbrace{H_2^e}_{=0} V \right]_0 \quad (56)$$

Or

$$\boxed{\int_0^{L^e} \frac{d^2H_2^e}{dx^2} EI \frac{d^2W}{dx^2} dx = \int_0^{L^e} H_2^e Q(x)\, dx + M_l^e} \tag{57}$$

Where M_l^e is the moment on the left end of the element.

Our third equation may be written as,

$$\int_0^{L^e} \frac{d^2H_3^e}{dx^2} EI \frac{d^2W}{dx^2} dx = \int_0^{L^e} H_3^e Q(x)\, dx - \left[\underbrace{\frac{dH_3^e}{dx}}_{=0} M\right]_{L^e} + \left[\underbrace{\frac{dH_3^e}{dx}}_{=0} M\right]_0 + \left[\underbrace{H_3^e}_{=1} V\right]_{L^e} - \left[\underbrace{H_3^e}_{=0} V\right]_0 \tag{58}$$

Or

$$\boxed{\int_0^{L^e} \frac{d^2H_3^e}{dx^2} EI \frac{d^2W}{dx^2} dx = \int_0^{L^e} H_3^e Q(x)\, dx + V_r^e} \tag{59}$$

Where M_l^e is the moment on the left end of the element.

Our fourth and final equation may be written as,

$$\int_0^{L^e} \frac{d^2H_4^e}{dx^2} EI \frac{d^2W}{dx^2} dx = \int_0^{L^e} H_4^e Q(x)\, dx - \left[\underbrace{\frac{dH_4^e}{dx}}_{=1} M\right]_{L^e} + \left[\underbrace{\frac{dH_4^e}{dx}}_{=0} M\right]_0 + \left[\underbrace{H_4^e}_{=0} V\right]_{L^e} - \left[\underbrace{H_4^e}_{=0} V\right]_0 \tag{60}$$

Or

$$\boxed{\int_0^{L^e} \frac{d^2H_4^e}{dx^2} EI \frac{d^2W}{dx^2} dx = \int_0^{L^e} H_4^e Q(x)\, dx - M_r^e} \tag{61}$$

Where M_l^e is the moment on the left end of the element.

We now begin to investigate the right hand side of these equations,

$$\int_0^{L^e} \frac{d^2 H_i^e}{dx^2} EI \frac{d^2 W}{dx^2} dx \tag{62}$$

Where,

$$W(x) = H_1^e(x) W_1^e + H_2^e(x) \theta_1^e + H_3^e(x) W_2^e + H_4^e(x) \theta_2^e \tag{63}$$

Or we could write this equation, only for convenience, as,

$$W(x) = \sum_{j=1}^{4} H_j^e(x) \Phi_j^e = H_1^e(x) \Phi_1^e + H_2^e(x) \Phi_2^e + H_3^e(x) \Phi_3^e + H_4^e(x) \Phi_4^e \tag{64}$$

This means our constants, Φ_i^e are actually,

$$\begin{Bmatrix} \Phi_1^e \\ \Phi_2^e \\ \Phi_3^e \\ \Phi_4^e \end{Bmatrix} = \begin{Bmatrix} W_1^e \\ \theta_1^e \\ W_2^e \\ \theta_2^e \end{Bmatrix} \tag{65}$$

We now insert our equation for the beam displacement into our weighted residual equation,

$$\int_0^{L^e} \frac{d^2 H_i^e}{dx^2} EI \frac{d^2 \left[\sum_{j=1}^{4} H_j^e \Phi_j^e\right]}{dx^2} dx \tag{66}$$

Moving the summation and our Φ_j^e term out from under the differential we get, NOTE: our Φ_j^es are constants, unknowns.

$$\int_0^{L^e} \frac{d^2 H_i^e}{dx^2} EI \left[\sum_{j=1}^{4} \frac{d^2 H_j^e}{dx^2} \Phi_j^e\right] dx \tag{67}$$

We may now move the summation and our Φ_j^e term out from under the integration,

$$\sum_{j=1}^{4} \left[\int_0^{L^e} \frac{d^2 H_i^e}{dx^2} EI \frac{d^2 H_j^e}{dx^2} dx\right] \Phi_j^e \tag{68}$$

This equation may be written as,

$$\sum_{j=1}^{4} [K_{ij}]\, \Phi_j^e \tag{69}$$

Where,

$$K_{ij} = \int_0^{L^e} \frac{d^2 H_i^e}{dx^2} EI \frac{d^2 H_j^e}{dx^2} dx \tag{70}$$

Carrying out all of our integrations using Maxima we get,

$$[K] = EI \begin{bmatrix} \frac{12}{L^3} & \frac{6}{L^2} & -\frac{12}{L^3} & \frac{6}{L^2} \\ \frac{6}{L^2} & \frac{4}{L} & -\frac{6}{L^2} & \frac{2}{L} \\ -\frac{12}{L^3} & -\frac{6}{L^2} & \frac{12}{L^3} & -\frac{6}{L^2} \\ \frac{6}{L^2} & \frac{2}{L} & -\frac{6}{L^2} & \frac{4}{L} \end{bmatrix} \tag{71}$$

This equation may be solved symbolically in Maxima by recognizing our matrix equation mat be generate from,

$$[K] = \int_0^{L^e} EI \begin{Bmatrix} \frac{d^2 H_1^e}{dx^2} \\ \frac{d^2 H_2^e}{dx^2} \\ \frac{d^2 H_3^e}{dx^2} \\ \frac{d^2 H_4^e}{dx^2} \end{Bmatrix} \begin{Bmatrix} \frac{d^2 H_1^e}{dx^2} & \frac{d^2 H_2^e}{dx^2} & \frac{d^2 H_3^e}{dx^2} & \frac{d^2 H_4^e}{dx^2} \end{Bmatrix} dx \tag{72}$$

This equation is often written as,

$$\underbrace{[K]}_{4\times 4} = \int_0^{L^e} \underbrace{[B]^T}_{4\times 1} \underbrace{[D]}_{1\times 1} \underbrace{[B]}_{1\times 4} dx \tag{73}$$

Where,

$$[B]^T = \begin{Bmatrix} \frac{d^2 H_1^e}{dx^2} \\ \frac{d^2 H_2^e}{dx^2} \\ \frac{d^2 H_3^e}{dx^2} \\ \frac{d^2 H_4^e}{dx^2} \end{Bmatrix} \qquad \text{And} \qquad [D] = EI \tag{74}$$

We now look at the distributed load $Q(x)$. The right hand side of the equation dealing with distributed load is,

$$F_i = \int_0^{L^e} H_i^e Q(x)\, dx \tag{75}$$

Which may be rewritten as,

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} = \int_0^{L^e} \begin{Bmatrix} H_1^e \\ H_2^e \\ H_3^e \\ H_4^e \end{Bmatrix} Q(x)\, dx \tag{76}$$

Now the question is, What type of distributed loads are we interested in. We shall limit ourself to 2 types, Constant load

$$Q(x) = q_0 \tag{77}$$

and linearly distributed loads,

$$Q(x) = q_1 x + q_0 \tag{78}$$

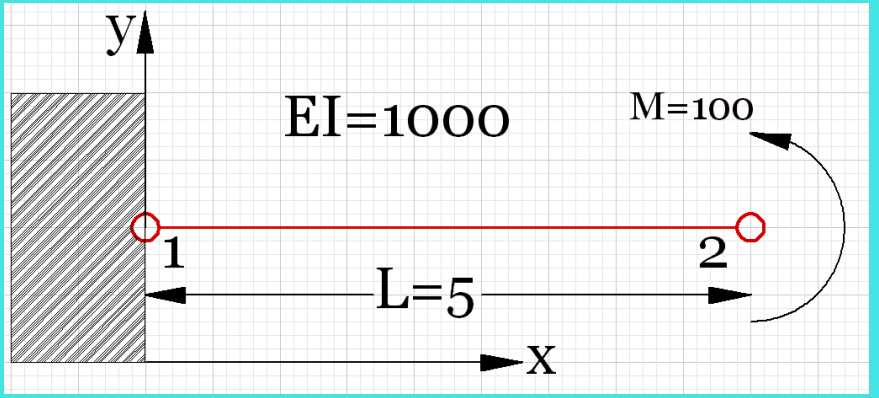
However, the linearly varying load can reduce down to the constant load. So we shall only consider the linearly varying load. Note, q_0 and q_1 are constants. For linear varying distributed loads we get,

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} = q_1 \begin{Bmatrix} \frac{3L^2}{20} \\ \frac{L^3}{30} \\ \frac{7L^2}{20} \\ -\frac{L^3}{20} \end{Bmatrix} + q_0 \begin{Bmatrix} \frac{L}{2} \\ \frac{L^2}{12} \\ \frac{L}{2} \\ -\frac{L^2}{12} \end{Bmatrix} \tag{79}$$

As a result of all of these calculations our element level matrix becomes,

$$\boxed{EI \begin{bmatrix} \frac{12}{L^3} & \frac{6}{L^2} & -\frac{12}{L^3} & \frac{6}{L^2} \\ \frac{6}{L^2} & \frac{4}{L} & -\frac{6}{L^2} & \frac{2}{L} \\ -\frac{12}{L^3} & -\frac{6}{L^2} & \frac{12}{L^3} & -\frac{6}{L^2} \\ \frac{6}{L^2} & \frac{2}{L} & -\frac{6}{L^2} & \frac{4}{L} \end{bmatrix} \begin{Bmatrix} W_1^e \\ \theta_1^e \\ W_2^e \\ \theta_2^e \end{Bmatrix} = q_1 \begin{Bmatrix} \frac{3L^2}{20} \\ \frac{L^3}{30} \\ \frac{7L^2}{20} \\ -\frac{L^3}{20} \end{Bmatrix} + q_0 \begin{Bmatrix} \frac{L}{2} \\ \frac{L^2}{12} \\ \frac{L}{2} \\ -\frac{L^2}{12} \end{Bmatrix} + \begin{Bmatrix} -V_l^e \\ M_l^e \\ V_r^e \\ -M_r^e \end{Bmatrix}} \tag{80}$$

This Is our all important system equation! We shall now process to several examples! We will begin with a one element beam as shown below,



Using all of our constants we obtain a matrix,

$$\begin{bmatrix} 96 & 240 & -96 & 240 \\ 240 & 800 & -240 & 400 \\ -96 & -240 & 96 & -240 \\ 240 & 400 & -240 & 800 \end{bmatrix} \begin{Bmatrix} W_1^e \\ \theta_1^e \\ W_2^e \\ \theta_2^e \end{Bmatrix} = \begin{Bmatrix} -V_l^e \\ M_l^e \\ V_r^e \\ -M_r^e \end{Bmatrix} \tag{81}$$

We first notice the left end is cantilevered.

$$\begin{aligned} W_1 &= 0 \\ \theta_1 &= 0 \end{aligned} \tag{82}$$

We also notice no shear load on the right hand side. In addition, we have a Moment of 100 ccw! We notice that this corresponds to a negative moment with respect to our definition of the moment on the right end of the beam. Hence,

$$\begin{aligned} V_r^e &= 0 \\ -M_r^e &= 100 \end{aligned} \tag{83}$$

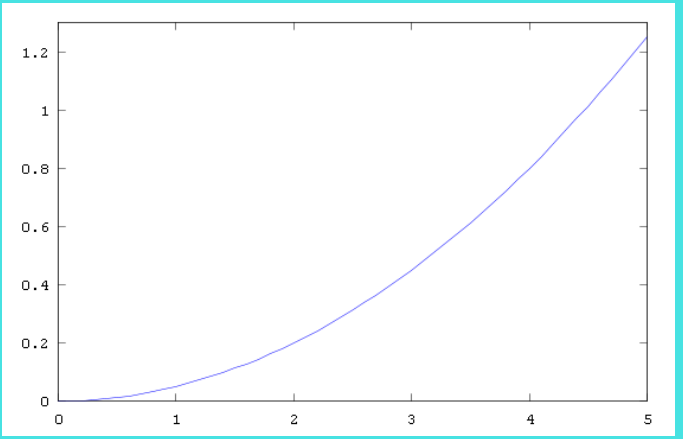
Hence our equation becomes,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -96 & -240 & 96 & -240 \\ 240 & 400 & -240 & 800 \end{bmatrix} \begin{Bmatrix} W_1^e \\ \theta_1^e \\ W_2^e \\ \theta_2^e \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 100 \end{Bmatrix} \tag{84}$$

Our solution to this equation becomes,

$$\begin{Bmatrix} W_1^e \\ \theta_1^e \\ W_2^e \\ \theta_2^e \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 1.25 \\ 0.5 \end{Bmatrix} \tag{85}$$

The deflection is shown below.



If we now reinsert our deflection into the Original Matrix,

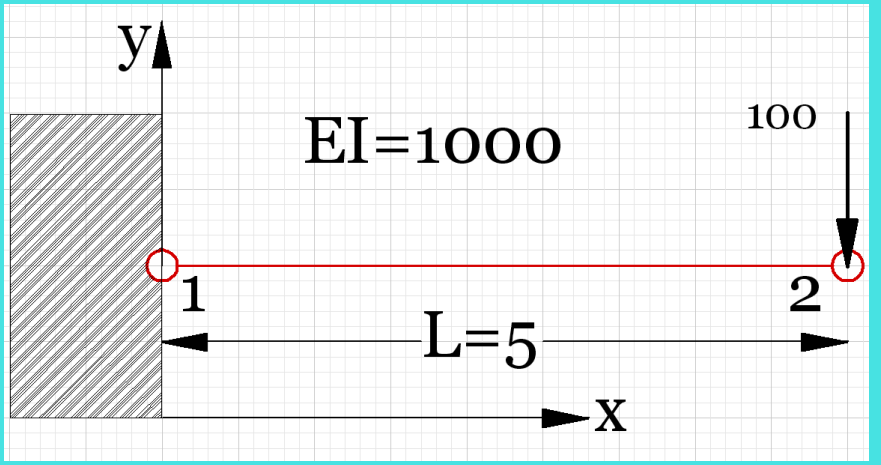
$$\begin{bmatrix} 96 & 240 & -96 & 240 \\ 240 & 800 & -240 & 400 \\ -96 & -240 & 96 & -240 \\ 240 & 400 & -240 & 800 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 1.25 \\ 0.5 \end{Bmatrix} = \begin{Bmatrix} -V_l^e \\ M_l^e \\ V_r^e \\ -M_r^e \end{Bmatrix} \tag{86}$$

Carrying out the matrix multiplication we get,

$$\begin{Bmatrix} -V_l^e \\ M_l^e \\ V_r^e \\ -M_r^e \end{Bmatrix} = \begin{Bmatrix} 0 \\ -100 \\ 0 \\ 100 \end{Bmatrix} \tag{87}$$

Which according to our sign convention the moment on the left end is clockwise and the moment on the right end is counterclockwise.

2.2. Experiment 2



Using all of our constants we obtain a matrix,

$$\begin{bmatrix} 96 & 240 & -96 & 240 \\ 240 & 800 & -240 & 400 \\ -96 & -240 & 96 & -240 \\ 240 & 400 & -240 & 800 \end{bmatrix} \begin{Bmatrix} W_1^e \\ \theta_1^e \\ W_2^e \\ \theta_2^e \end{Bmatrix} = \begin{Bmatrix} -V_l^e \\ M_l^e \\ V_r^e \\ -M_r^e \end{Bmatrix} \tag{88}$$

We first notice the left end is cantilevered.

$$\begin{aligned} W_1 &= 0 \\ \theta_1 &= 0 \end{aligned} \tag{89}$$

We also notice no moment on the right hand side. In addition, we have a shear load of 100! We notice that this corresponds to a negative shear with respect to our definition of the shear on the right end of the beam. Hence,

$$\begin{aligned} V_r^e &= -100 \\ -M_r^e &= 0 \end{aligned} \tag{90}$$

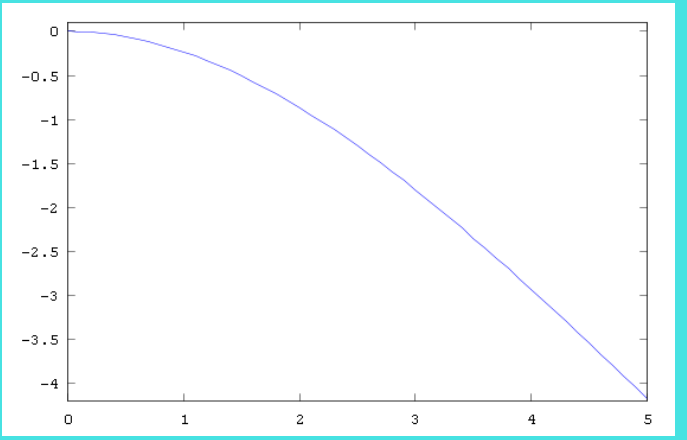
Hence our equation becomes,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -96 & -240 & 96 & -240 \\ 240 & 400 & -240 & 800 \end{bmatrix} \begin{Bmatrix} W_1^e \\ \theta_1^e \\ W_2^e \\ \theta_2^e \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -100 \\ 0 \end{Bmatrix} \tag{91}$$

Our solution to this equation becomes,

$$\begin{Bmatrix} W_1^e \\ \theta_1^e \\ W_2^e \\ \theta_2^e \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -4.1667 \\ -1.25 \end{Bmatrix} \tag{92}$$

The deflection is shown below.



If we now reinsert our deflection into the Original Matrix,

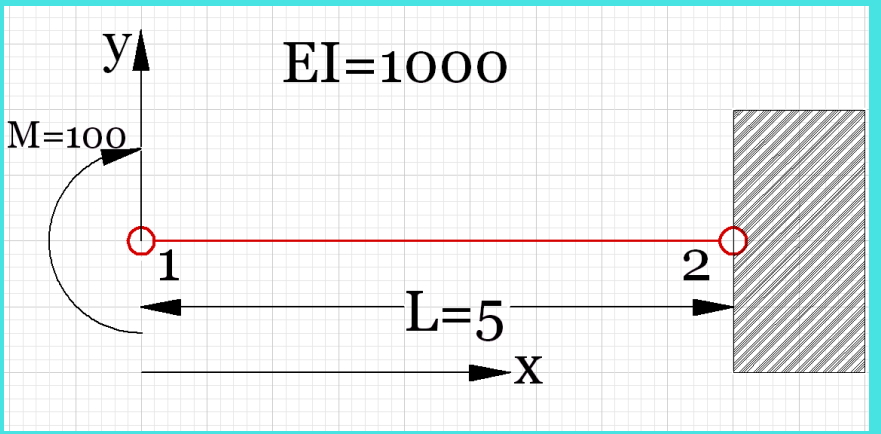
$$\begin{bmatrix} 96 & 240 & -96 & 240 \\ 240 & 800 & -240 & 400 \\ -96 & -240 & 96 & -240 \\ 240 & 400 & -240 & 800 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -4.1667 \\ -1.25 \end{Bmatrix} = \begin{Bmatrix} -V_l^e \\ M_l^e \\ V_r^e \\ -M_r^e \end{Bmatrix} \tag{93}$$

Carrying out the matrix multiplication we get,

$$\begin{Bmatrix} -V_l^e \\ M_l^e \\ V_r^e \\ -M_r^e \end{Bmatrix} = \begin{Bmatrix} 100 \\ 500 \\ -100 \\ 0 \end{Bmatrix} \tag{94}$$

Which according to our sign convention the moment on the left end is clockwise 500. Which corresponds to our 100 load a distance of 5 down the beam. The shear on the left end of the beam is opposite to the applied load on the right end. The left end shear is in the positive y direction. The shear on the right end of the beam is in the negative y direction. Finally, there is no moment on the right end of the beam.

2.3. Experiment 3



Using all of our constants we obtain a matrix,

$$\begin{bmatrix} 96 & 240 & -96 & 240 \\ 240 & 800 & -240 & 400 \\ -96 & -240 & 96 & -240 \\ 240 & 400 & -240 & 800 \end{bmatrix} \begin{Bmatrix} W_1^e \\ \theta_1^e \\ W_2^e \\ \theta_2^e \end{Bmatrix} = \begin{Bmatrix} -V_l^e \\ M_l^e \\ V_r^e \\ -M_r^e \end{Bmatrix} \tag{95}$$

We first notice the right end is cantilevered.

$$\begin{aligned} W_2 &= 0 \\ \theta_2 &= 0 \end{aligned} \tag{96}$$

We also notice no shear on the left hand side. In addition, we have a moment load of 100 cw! We notice that this corresponds to a negative moment with respect to our definition of the moment on the left end of the beam. Hence,

$$\begin{aligned} V_l^e &= 0 \\ M_l^e &= -100 \end{aligned} \tag{97}$$

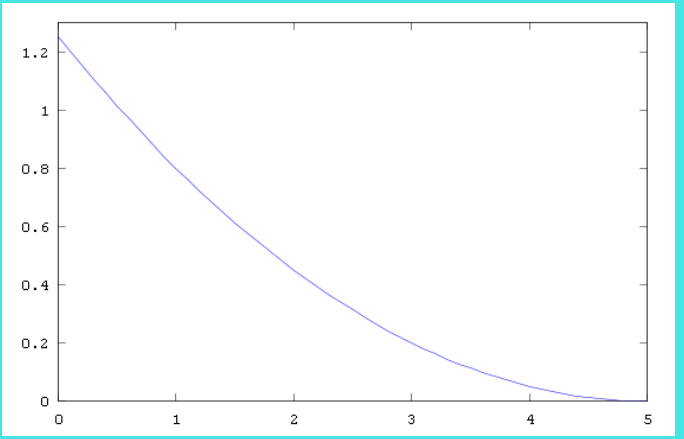
Hence our equation becomes,

$$\begin{bmatrix} 96 & 240 & -96 & 240 \\ 240 & 800 & -240 & 400 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} W_1^e \\ \theta_1^e \\ W_2^e \\ \theta_2^e \end{Bmatrix} = \begin{Bmatrix} 0 \\ -100 \\ 0 \\ 0 \end{Bmatrix} \tag{98}$$

Our solution to this equation becomes,

$$\begin{Bmatrix} W_1^e \\ \theta_1^e \\ W_2^e \\ \theta_2^e \end{Bmatrix} = \begin{Bmatrix} 1.25 \\ -.5 \\ 0 \\ 0 \end{Bmatrix} \tag{99}$$

The deflection is shown below.



If we now reinsert our deflection into the Original Matrix,

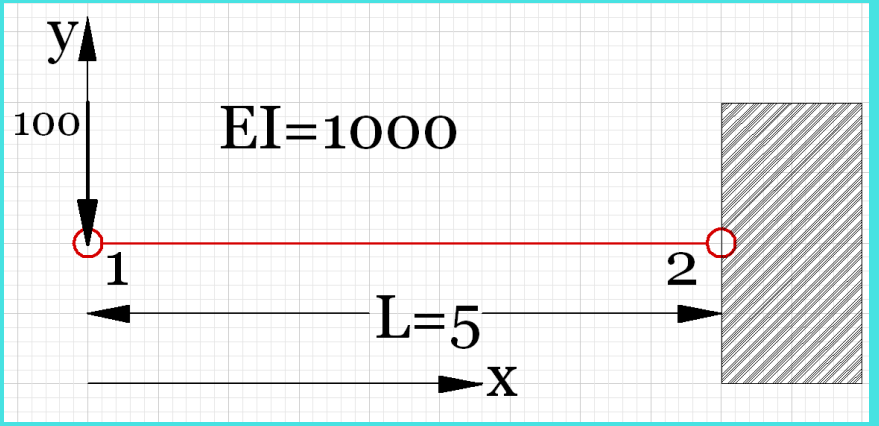
$$\begin{bmatrix} 96 & 240 & -96 & 240 \\ 240 & 800 & -240 & 400 \\ -96 & -240 & 96 & -240 \\ 240 & 400 & -240 & 800 \end{bmatrix} \begin{Bmatrix} 1.25 \\ -.5 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -V_l^e \\ M_l^e \\ V_r^e \\ -M_r^e \end{Bmatrix} \tag{100}$$

Carrying out the matrix multiplication we get,

$$\begin{Bmatrix} -V_l^e \\ M_l^e \\ V_r^e \\ -M_r^e \end{Bmatrix} = \begin{Bmatrix} 0 \\ -100 \\ 0 \\ 100 \end{Bmatrix} \tag{101}$$

Which according to our sign convention the moment on the left end is clockwise 100. The shear on the left end of the beam is zero. The shear on the right end of the beam is zero as well. The moment on the right end of the beam is counterclockwise 100.

2.4. Experiment 4



Using all of our constants we obtain a matrix,

$$\begin{bmatrix} 96 & 240 & -96 & 240 \\ 240 & 800 & -240 & 400 \\ -96 & -240 & 96 & -240 \\ 240 & 400 & -240 & 800 \end{bmatrix} \begin{Bmatrix} W_1^e \\ \theta_1^e \\ W_2^e \\ \theta_2^e \end{Bmatrix} = \begin{Bmatrix} -V_l^e \\ M_l^e \\ V_r^e \\ -M_r^e \end{Bmatrix} \tag{102}$$

We first notice the right end is cantilevered.

$$\begin{aligned} W_2 &= 0 \\ \theta_2 &= 0 \end{aligned} \tag{103}$$

We also notice no moment on the left hand side. In addition, we have a shear load of 100 in the negative coordinate direction! We notice that this corresponds to a negative shear with respect to our definition of the shear on the left end of the beam. Hence,

$$\begin{aligned} V_l^e &= -100 \\ M_l^e &= 0 \end{aligned} \tag{104}$$

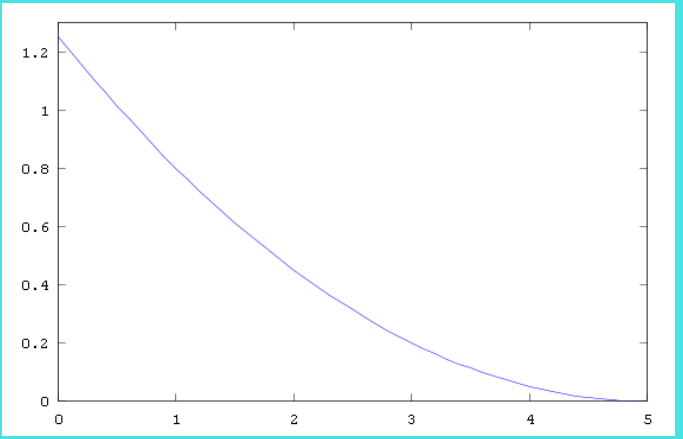
Hence our equation becomes,

$$\begin{bmatrix} 96 & 240 & -96 & 240 \\ 240 & 800 & -240 & 400 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} W_1^e \\ \theta_1^e \\ W_2^e \\ \theta_2^e \end{Bmatrix} = \begin{Bmatrix} -100 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{105}$$

Our solution to this equation becomes,

$$\begin{Bmatrix} W_1^e \\ \theta_1^e \\ W_2^e \\ \theta_2^e \end{Bmatrix} = \begin{Bmatrix} 1.25 \\ -.5 \\ 0 \\ 0 \end{Bmatrix} \tag{106}$$

The deflection is shown below.



If we now reinsert our deflection into the Original Matrix,

$$\begin{bmatrix} 96 & 240 & -96 & 240 \\ 240 & 800 & -240 & 400 \\ -96 & -240 & 96 & -240 \\ 240 & 400 & -240 & 800 \end{bmatrix} \begin{Bmatrix} 1.25 \\ -.5 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -V_l^e \\ M_l^e \\ V_r^e \\ -M_r^e \end{Bmatrix} \tag{107}$$

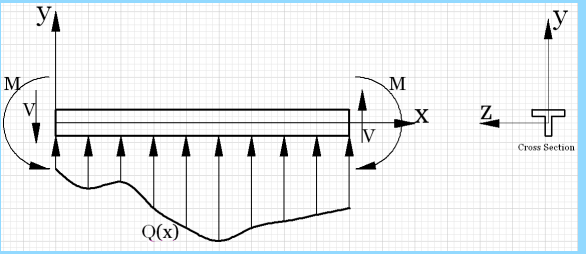
Carrying out the matrix multiplication we get,

$$\begin{Bmatrix} -V_l^e \\ M_l^e \\ V_r^e \\ -M_r^e \end{Bmatrix} = \begin{Bmatrix} 0 \\ -100 \\ 0 \\ 100 \end{Bmatrix} \tag{108}$$

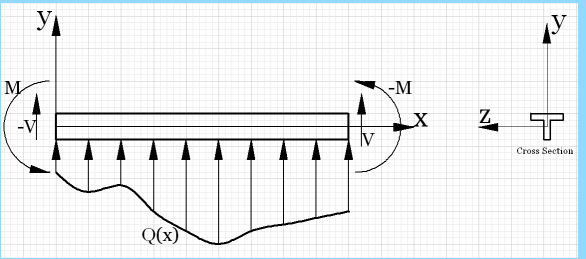
Which according to our sign convention the moment on the left end is clockwise 100. The shear on the left end of the beam is zero. The shear on the right end of the beam is zero as well. The moment on the right end of the beam is counterclockwise 100.

3. Sign Convention MoM vs. FEM

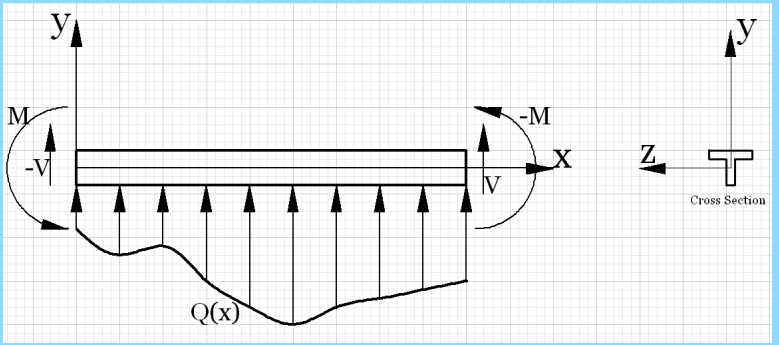
At this point it may be instructive to glance back at our sign convention for mechanics of materials (MoM). we have seen this figure previously,

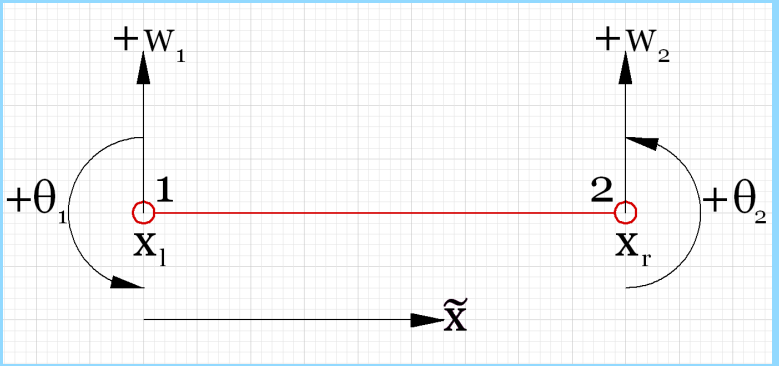


This plot shows the standard sign convention/ directions for our shear and moments. These definitions are used in the derivation of the beams equations in mechanics of materials. If we change the direction of the shear on the left end of our beam and change direction of our moment on the right end of the beam we obtain,

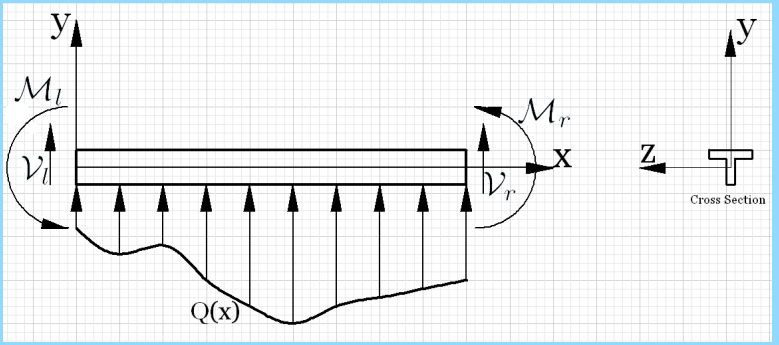


If we compare this plot to that of our displacement and rotations defined in the finite element method (FEM) we see,





If we redefine our shear and moment directions as well as notation we see a consistent sign convention as,



Where,

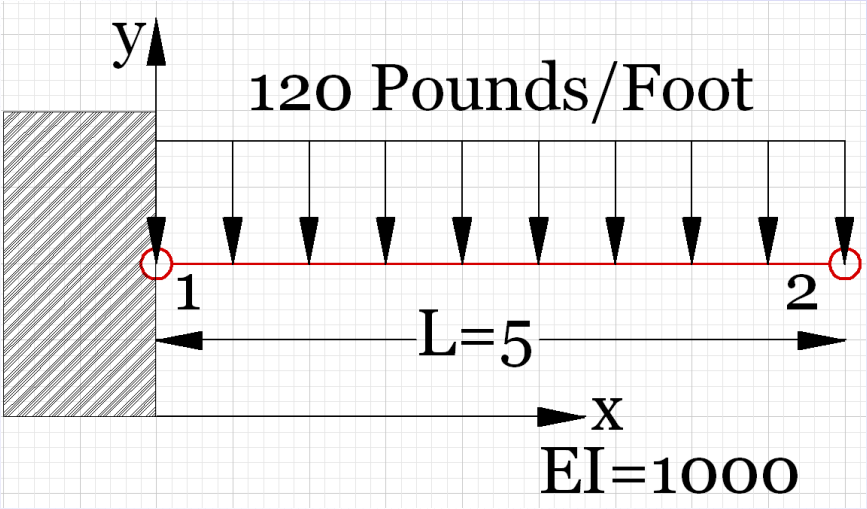
- $\mathcal{V}_l \Longrightarrow$ is the shear on the left end of the beam (FEM) notation.
- $\mathcal{V}_r \Longrightarrow$ is the shear on the right end of the beam (FEM) notation.
- $\mathcal{M}_l \Longrightarrow$ is the moment on the left end of the beam (FEM) notation.
- $\mathcal{M}_r \Longrightarrow$ is the moment on the right end of the beam (FEM) notation.

With this our finite element matrix becomes,

$$EI\begin{bmatrix} \frac{12}{L^3} & \frac{6}{L^2} & -\frac{12}{L^3} & \frac{6}{L^2} \\ \frac{6}{L^2} & \frac{4}{L} & -\frac{6}{L^2} & \frac{2}{L} \\ -\frac{12}{L^3} & -\frac{6}{L^2} & \frac{12}{L^3} & -\frac{6}{L^2} \\ \frac{6}{L^2} & \frac{2}{L} & -\frac{6}{L^2} & \frac{4}{L} \end{bmatrix}\begin{Bmatrix} W_1^e \\ \theta_1^e \\ W_2^e \\ \theta_2^e \end{Bmatrix} = q_1\begin{Bmatrix} \frac{3L^2}{20} \\ \frac{L^3}{30} \\ \frac{7L^2}{20} \\ -\frac{L^3}{20} \end{Bmatrix} + q_0\begin{Bmatrix} \frac{L}{2} \\ \frac{L^2}{12} \\ \frac{L}{2} \\ -\frac{L^2}{12} \end{Bmatrix} + \begin{Bmatrix} \mathcal{V}_l^e \\ \mathcal{M}_l^e \\ \mathcal{V}_r^e \\ \mathcal{M}_r^e \end{Bmatrix} \tag{109}$$

NOTE: All *shear* and *deflection* is in the positive coordinate direction! All *rotations* and *moments* are positive counterclockwise. NOTE: All *rotations* and *moments* are positive $\hat{k}$ direction (Right Hand Rule, for moving to 3-D later). NOTE: $\hat{k} = \hat{i} \times \hat{j}$

4. Problem 1



Using all of our constants we obtain a matrix,

$$\begin{bmatrix} 96 & 240 & -96 & 240 \\ 240 & 800 & -240 & 400 \\ -96 & -240 & 96 & -240 \\ 240 & 400 & -240 & 800 \end{bmatrix} \begin{Bmatrix} W_1^e \\ \theta_1^e \\ W_2^e \\ \theta_2^e \end{Bmatrix} = \begin{Bmatrix} -300 \\ -250 \\ -300 \\ 250 \end{Bmatrix} + \begin{Bmatrix} -V_l^e \\ M_l^e \\ V_r^e \\ -M_r^e \end{Bmatrix} = \begin{Bmatrix} -300 \\ -250 \\ -300 \\ 250 \end{Bmatrix} + \begin{Bmatrix} \mathcal{V}_l^e \\ \mathcal{M}_l^e \\ \mathcal{V}_r^e \\ \mathcal{M}_r^e \end{Bmatrix} \tag{110}$$

At this point our matrix equation is singular. This signifies there can be no solution without the proper boundary conditions. Our drawing indicate a cantilevered beam,

$$W_1^e = 0 \tag{111}$$

and

$$\theta_1^e = 0 \tag{112}$$

Hence we can simply replace these equations in our matrix with our boundary conditions,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -96 & -240 & 96 & -240 \\ 240 & 400 & -240 & 800 \end{bmatrix} \begin{Bmatrix} W_1^e \\ \theta_1^e \\ W_2^e \\ \theta_2^e \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -300 \\ 250 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ \mathcal{V}_r^e \\ \mathcal{M}_r^e \end{Bmatrix} \tag{113}$$

Now we have two hanging terms on the right hand side, V_r^e and M_r^e. Since there is no applied load or moment on the right hand side of our beam Both of these terms are zero,

$$\mathcal{V}_r^e = 0 \tag{114}$$

$$\mathcal{M}_r^e = 0 \tag{115}$$

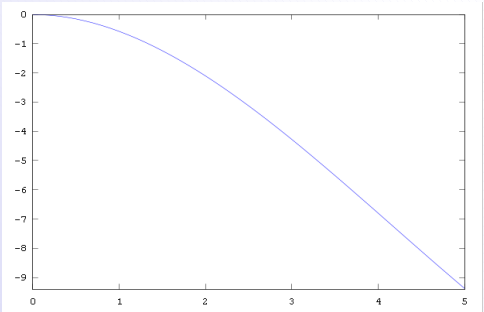
Hence our equations become,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -96 & -240 & 96 & -240 \\ 240 & 400 & -240 & 800 \end{bmatrix} \begin{Bmatrix} W_1^e \\ \theta_1^e \\ W_2^e \\ \theta_2^e \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -300 \\ 250 \end{Bmatrix} \tag{116}$$

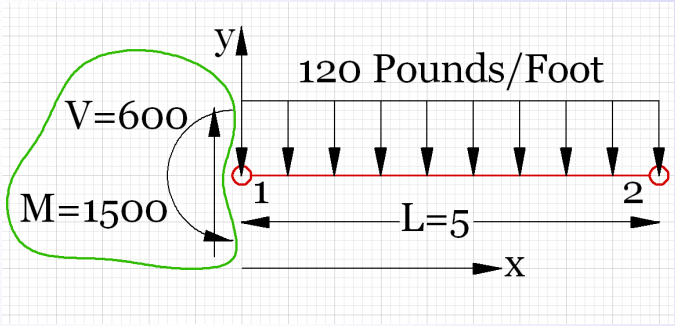
The solution to this matrix system is,

$$\begin{Bmatrix} W_1^e \\ \theta_1^e \\ W_2^e \\ \theta_2^e \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -9.375 \\ -2.5 \end{Bmatrix} \tag{117}$$

A plot of the deflection is shown below,



We know from mechanics of materials that our moment and shear on the left end of our beam is,



If we go back to our original equation and place our displacements into this equation,

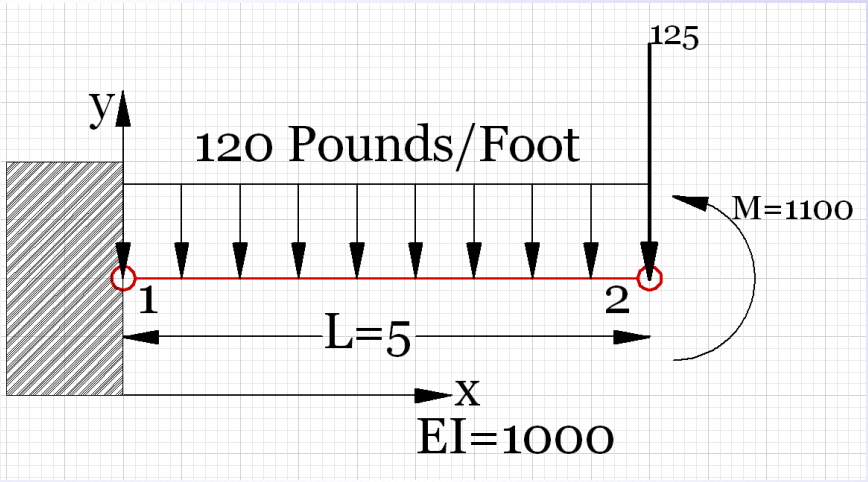
$$\begin{bmatrix} 96 & 240 & -96 & 240 \\ 240 & 800 & -240 & 400 \\ -96 & -240 & 96 & -240 \\ 240 & 400 & -240 & 800 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -9.375 \\ -2.5 \end{Bmatrix} = \begin{Bmatrix} -300 \\ -250 \\ -300 \\ 250 \end{Bmatrix} + \begin{Bmatrix} -V_l^e \\ M_l^e \\ V_r^e \\ -M_r^e \end{Bmatrix} = \begin{Bmatrix} -300 \\ -250 \\ -300 \\ 250 \end{Bmatrix} + \begin{Bmatrix} \mathcal{V}_l^e \\ \mathcal{M}_l^e \\ \mathcal{V}_r^e \\ \mathcal{M}_r^e \end{Bmatrix} \tag{118}$$

We get,

$$\begin{Bmatrix} \mathcal{V}_l^e \\ \mathcal{M}_l^e \\ \mathcal{V}_r^e \\ \mathcal{M}_r^e \end{Bmatrix} = \begin{Bmatrix} -V_l^e \\ M_l^e \\ V_r^e \\ -M_r^e \end{Bmatrix} = \begin{Bmatrix} 600 \\ 1500 \\ 0 \\ 0 \end{Bmatrix} \tag{119}$$

According to our beam coordinate developed from Mechanics of Materials this is correct. Shear down on the left end of a beam section is positive while a counter clockwise moment on the left end a beam section is positive. Shear up on the right end a beam section is positive while a clockwise moment on the right end of a beam section is positive. However, it is much easier to remember shear and displacement is positive up (FEM coordinates / notation) and rotations and moments are always positive counterclockwise.

4.1. Problem 2



Using all of our constants we obtain a matrix,

$$\begin{bmatrix} 96 & 240 & -96 & 240 \\ 240 & 800 & -240 & 400 \\ -96 & -240 & 96 & -240 \\ 240 & 400 & -240 & 800 \end{bmatrix} \begin{Bmatrix} W_1 \\ \theta_1 \\ W_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} -300 \\ -250 \\ -300 \\ 250 \end{Bmatrix} + \begin{Bmatrix} -V_1 \\ M_1 \\ V_2 \\ -M_2 \end{Bmatrix} = \begin{Bmatrix} -300 \\ -250 \\ -300 \\ 250 \end{Bmatrix} + \begin{Bmatrix} \mathcal{V}_1 \\ \mathcal{M}_1 \\ \mathcal{V}_2 \\ \mathcal{M}_2 \end{Bmatrix} \tag{120}$$

Our Boundary Conditions are,

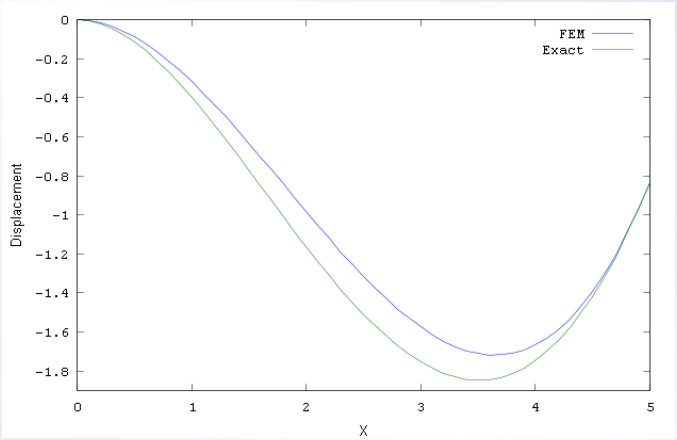
$$\begin{aligned} W_1 &= 0 \\ \theta_1 &= 0 \\ \mathcal{V}_2 &= -125 \\ \mathcal{M}_2 &= 1100 \end{aligned} \tag{121}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -96 & -240 & 96 & -240 \\ 240 & 400 & -240 & 800 \end{bmatrix} \begin{Bmatrix} W_1^e \\ \theta_1^e \\ W_2^e \\ \theta_2^e \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -300 \\ 250 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ -125 \\ 1100 \end{Bmatrix} \tag{122}$$

Inverting our equation, see Maxima, our solution becomes,

$$\begin{Bmatrix} W_1^e \\ \theta_1^e \\ W_2^e \\ \theta_2^e \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -0.8333333 \\ 1.4375 \end{Bmatrix} \tag{123}$$

Our deflection look like,



Our end loads come from our original equation,

$$\begin{bmatrix} 96 & 240 & -96 & 240 \\ 240 & 800 & -240 & 400 \\ -96 & -240 & 96 & -240 \\ 240 & 400 & -240 & 800 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -0.8333333 \\ 1.4375 \end{Bmatrix} = \begin{Bmatrix} -300 \\ -250 \\ -300 \\ 250 \end{Bmatrix} + \begin{Bmatrix} -V_1 \\ M_1 \\ V_2 \\ -M_2 \end{Bmatrix} = \begin{Bmatrix} -300 \\ -250 \\ -300 \\ 250 \end{Bmatrix} + \begin{Bmatrix} \mathcal{V}_1 \\ \mathcal{M}_1 \\ \mathcal{V}_2 \\ \mathcal{M}_2 \end{Bmatrix} \tag{124}$$

Which Yields (See Octave code),

$$\begin{Bmatrix} -V_1 \\ M_1 \\ V_2 \\ -M_2 \end{Bmatrix} = \begin{Bmatrix} \mathcal{V}_1 \\ \mathcal{M}_1 \\ \mathcal{V}_2 \\ \mathcal{M}_2 \end{Bmatrix} = \begin{Bmatrix} 725 \\ 1025 \\ -125 \\ 1100 \end{Bmatrix} \tag{125}$$

4.2. Shear and Moment Diagrams

It is common and very useful to create shear and moment diagrams in mechanics of material. From these we can determine the stresses in our beams. There are two ways of creating these diagrams within Finite Elements. The first is from the actual displacement distributions.

We will begin by observing the moment plots. Remembering our MoM definition of the moment,

$$M = -EI\frac{d^2W}{dx^2} \tag{126}$$

With this definition and our finite element equations we have,

$$M = -EI\sum_{i=1}^{4}\frac{d^2H_i^e}{dx^2}\Phi_i^e \tag{127}$$

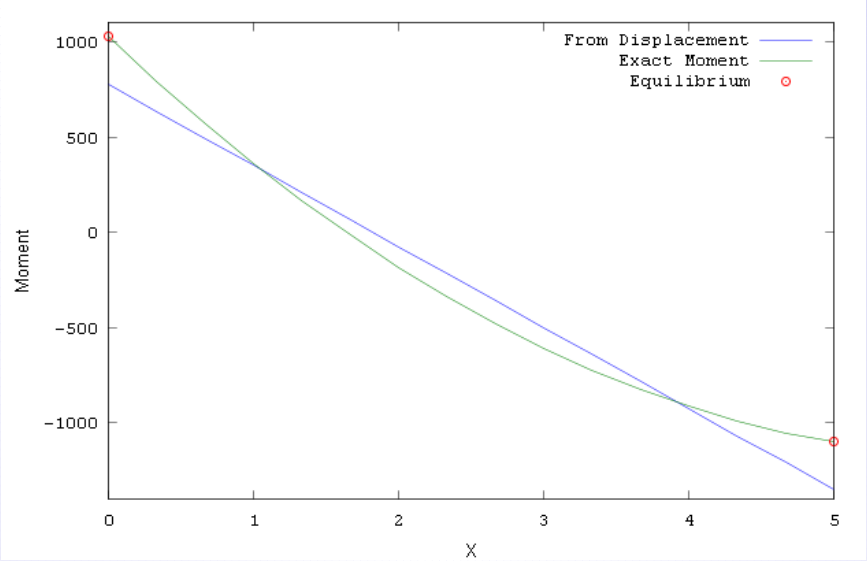
Carrying out the indicated differentiation we have,

$$M = -EI\left[\left(\frac{12x}{L^3}-\frac{6}{L^2}\right)W_1^e+\left(\frac{6x}{L^2}-\frac{4}{L}\right)\theta_1^e+\left(\frac{6}{L^2}-\frac{12x}{L^3}\right)W_2^e+\left(\frac{6x}{L^2}-\frac{2}{L}\right)\theta_2^e\right] \tag{128}$$

Which is a MOST a linear distribution! The shear equation is,

$$M = -EI\left[\frac{12}{L^3}W_1^e+\frac{6}{L^2}\theta_1^e-\frac{12}{L^3}W_2^e+\frac{6}{L^2}\theta_2^e\right] \tag{129}$$

Which is at most a Constant! If we decided to put our displacements into these equations, FROM OUR LAST CASE, Our plots would be,



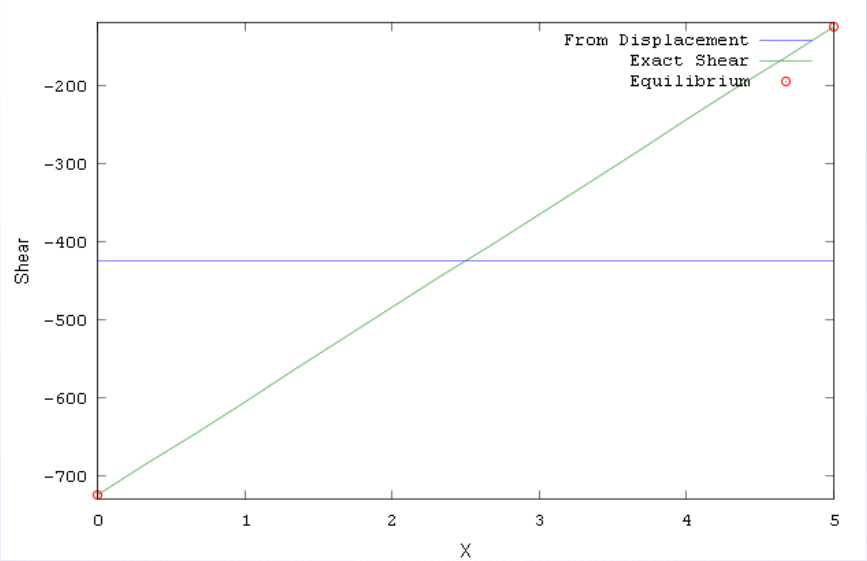
This is a problem! Since our equation is,

$$\frac{d^2}{dx^2}\left[EI\frac{d^2W}{dx^2}\right] = 120 \tag{130}$$

This means our exact solution is,

$$W(x) = -\frac{41x^2}{80} + \frac{29x^3}{240} - \frac{x^4}{200} \tag{131}$$

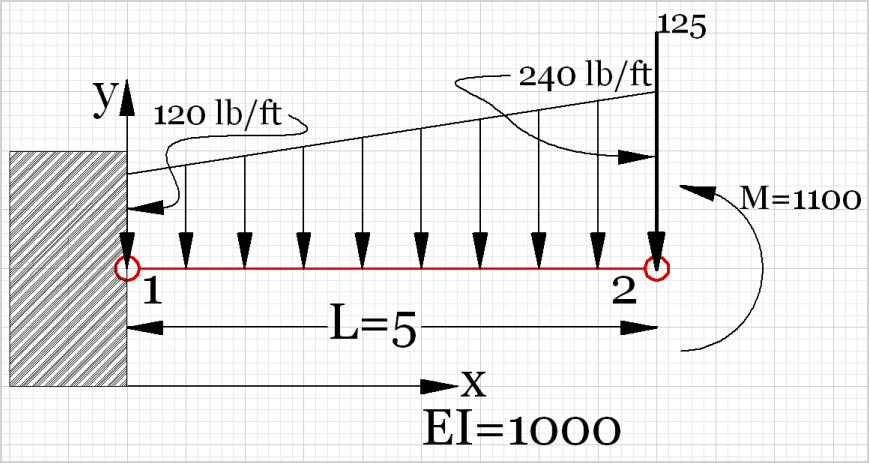
Which is 4^{th} order. There is a Problem here! Standard finite element solution iN all commercial software will only obtain the exact solution up to third order. This means ANY beam that has a distributed load on it will not be solved exactly. As a result the moment and shear is only available and accurate at the nodes. **WE WILL CHANGE THIS BY ADDING MORE DEGREES OF FREEDOM AND OUR SOFTWARE WILL OBTAIN THE EXACT SOLUTION FOR UP TO, LINEARLY VARYING DISTRIBUTED LOADS!.**


From Displacement
Exact Shear
Equilibrium
Shear
X
-200
-300
-400
-500
-600
-700
0
1
2
3
4
5

5. Problem 3

This problem will further expose weakness in standard finite element analysis of beams. We will use this problem as a test case to expand our Hermite functions and obtain the exact solution for linearly varying loads. We will first show a solution using standard FEM.

Our problem is,



Our distributed load is,

$$Q(x) = -120 - 24x \tag{132}$$

Our equation becomes,

$$1000 \begin{bmatrix} \frac{12}{L^3} & \frac{6}{L^2} & -\frac{12}{L^3} & \frac{6}{L^2} \\ \frac{6}{L^2} & \frac{4}{L} & -\frac{6}{L^2} & \frac{2}{L} \\ -\frac{12}{L^3} & -\frac{6}{L^2} & \frac{12}{L^3} & -\frac{6}{L^2} \\ \frac{6}{L^2} & \frac{2}{L} & -\frac{6}{L^2} & \frac{4}{L} \end{bmatrix} \begin{Bmatrix} W_1^e \\ \theta_1^e \\ W_2^e \\ \theta_2^e \end{Bmatrix} = -24 \begin{Bmatrix} \frac{3L^2}{20} \\ \frac{L^3}{30} \\ \frac{7L^2}{20} \\ -\frac{L^3}{20} \end{Bmatrix} - 120 \begin{Bmatrix} \frac{L}{2} \\ \frac{L^2}{12} \\ \frac{L}{2} \\ -\frac{L^2}{12} \end{Bmatrix} + \begin{Bmatrix} \mathcal{V}_l^e \\ \mathcal{M}_l^e \\ \mathcal{V}_r^e \\ \mathcal{M}_r^e \end{Bmatrix} \tag{133}$$

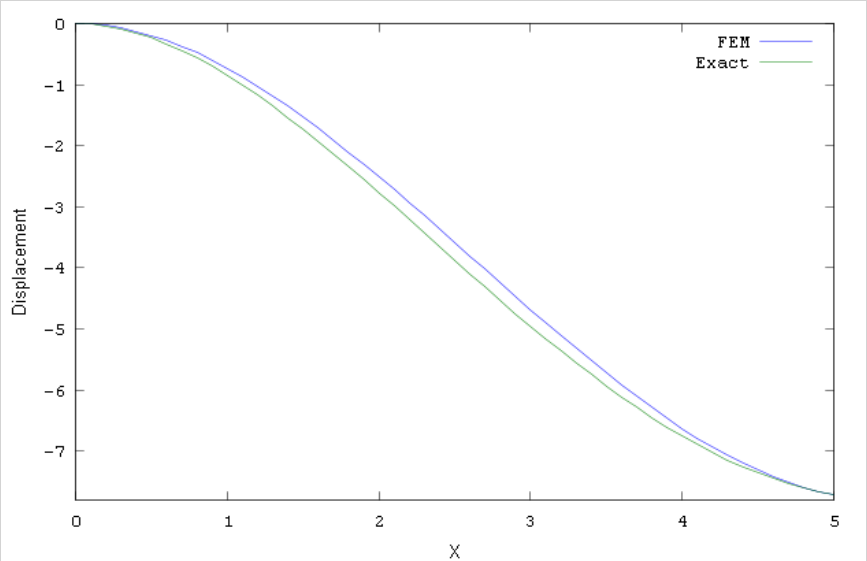
Again implementing our boundary conditions,

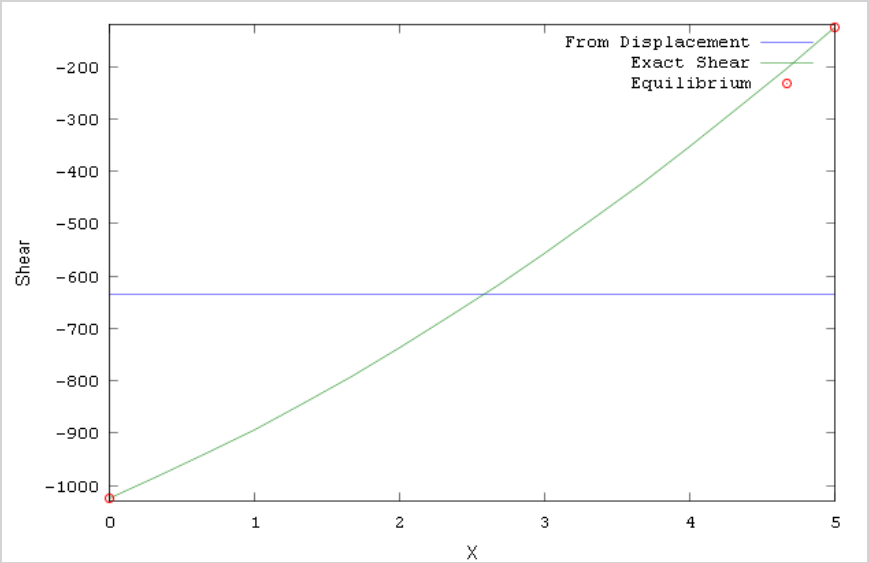
$$\begin{aligned} W_1 &= 0 \\ \theta_1 &= 0 \\ \mathcal{V}_2 &= -125 \\ \mathcal{M}_2 &= 1100 \end{aligned} \tag{134}$$

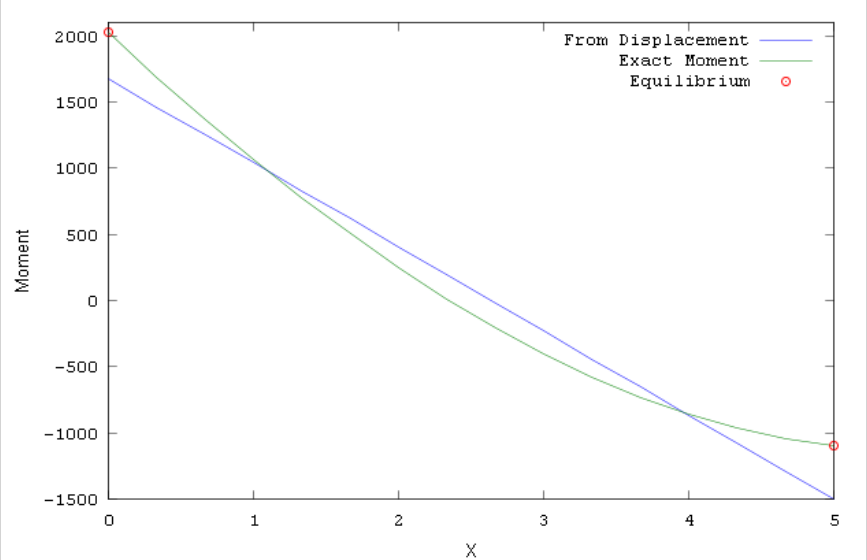
We get,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -96 & -240 & 96 & -240 \\ 240 & 400 & -240 & 800 \end{bmatrix} \begin{Bmatrix} W_1^e \\ \theta_1^e \\ W_2^e \\ \theta_2^e \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -510 \\ 400 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ -125 \\ 1100 \end{Bmatrix} \tag{135}$$

Which gives a solution of, (See Octave and Maxima),




From Displacement
Exact Shear
Equilibrium
Shear
X
-200
-300
-400
-500
-600
-700
-800
-900
-1000
0
1
2
3
4
5



Which seems reasonable, because the maximum moment is at the ends of the element. An in fact the moment gives the major component of the stress,

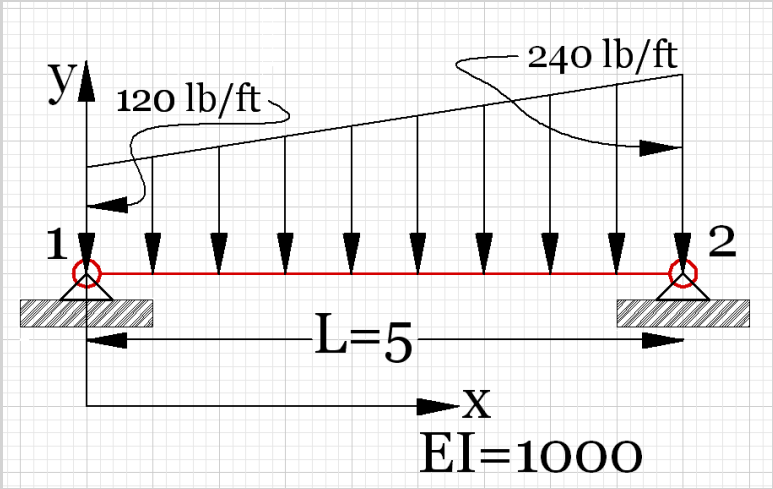
$$\sigma = \frac{Mc}{I} \tag{136}$$

But is this always the case?

NO.....

5.1. Problem 4

What about a simply supported beam?

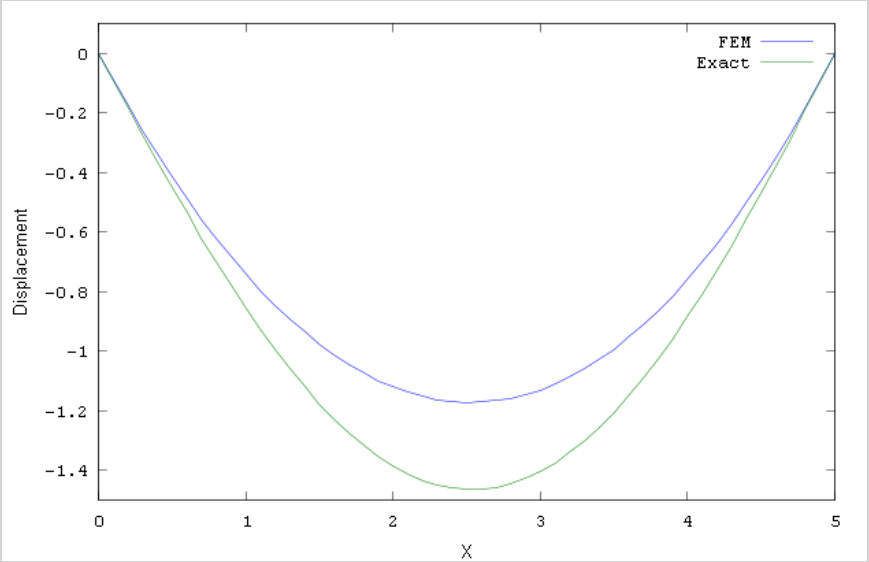


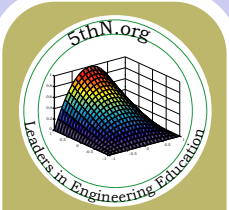
Our boundary conditions are,

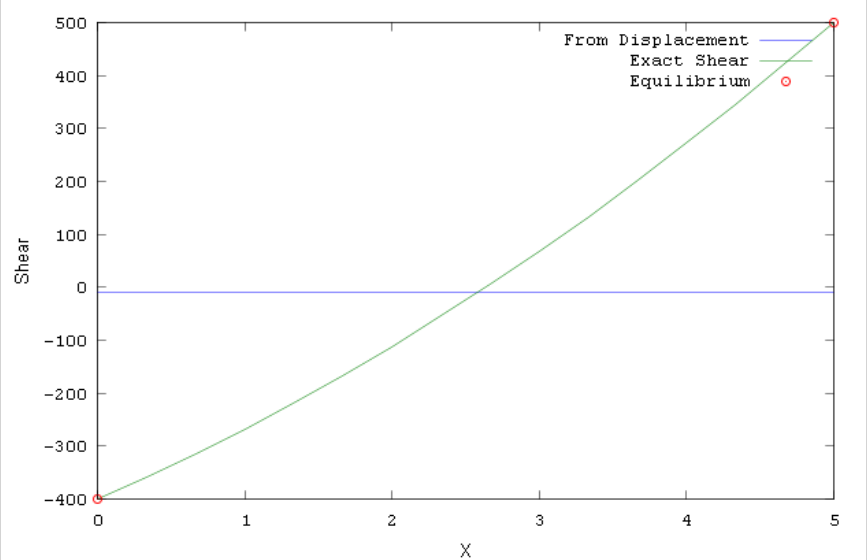
$$\begin{aligned} W_1 &= 0 \\ \mathcal{M}_1 &= 0 \\ W_2 &= 0 \\ \mathcal{M}_2 &= 0 \end{aligned} \tag{137}$$

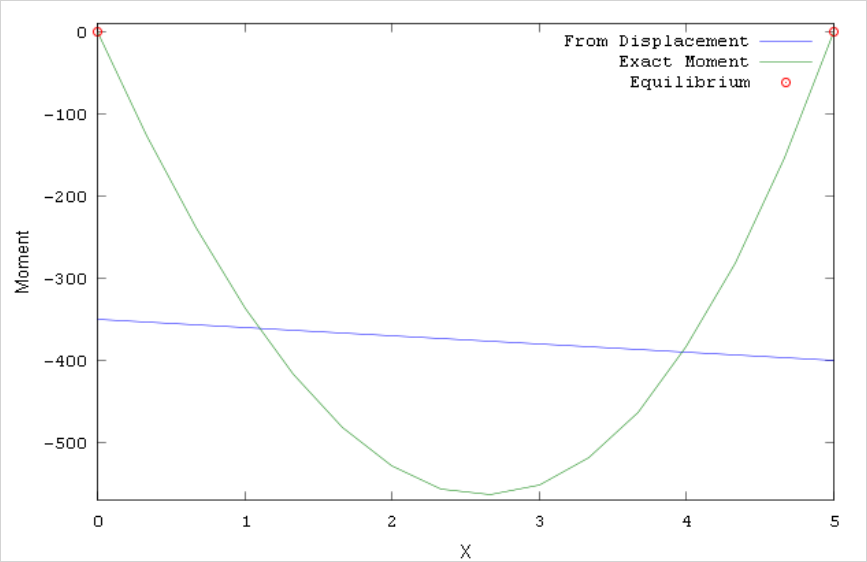
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 240 & 800 & -200 & 400 \\ 0 & 0 & 1 & 0 \\ 240 & 400 & -240 & 800 \end{bmatrix} \begin{Bmatrix} W_1^e \\ \theta_1^e \\ W_2^e \\ \theta_2^e \end{Bmatrix} = \begin{Bmatrix} 0 \\ -350 \\ 0 \\ 400 \end{Bmatrix} \tag{138}$$

Which gives a solution of, (See Octave and Maxima),


FEM
Exact
Displacement
X
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-1.4
0
1
2
3
4
5


5thN.org
Leaders in Engineering Education


From Displacement
Exact Shear
Equilibrium
Shear
X
500
400
300
200
100
0
-100
-200
-300
-400
0
1
2
3
4
5



This last plot, the moment, gives rise to a possible problem. Equilibrium shows a zero moment on either end of the beam. The displacement profile shows a moment from -400 at the right end to -350 on the left end, neither of which is correct. What is the maximum moment and therefore the maximum bending stress in the beam? The exact solution shows a moment -564. This value will produce a much larger stress than any finite element value. This is a real problem. In the next section we propose a new displacement distribution to deal with this.

6. Hermite Basis Functions

Several slides ago we introduced, with explanation, the standard Hermite cubic basis functions. These functions are used throughout all finite element textbooks and commercial codes. This is a problem, since we are commonly interested in obtaining an accurate solution for stresses in a beam. Many people will argue that including more and more elements in a solution will give a better approximation to the moment and therefore the stress. This is true. However, since we can obtain the exact solution for linear distributed loads on a beam with one element, it seems like a good idea to do so!

We begin by looking ate the standard Hermite cubic functions used throughout the finite element community. We begin by specifying a standard third order function,

$$F(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \tag{139}$$

We can generate our 4 basis functions by using this simple third order function, See Maxima File.

Forcing the following conditions on this function,

$$\begin{aligned} [F]_{x=0} &= 1 \\ \left[\frac{dF}{dx}\right]_{x=0} &= 0 \\ [F]_{x=L} &= 0 \\ \left[\frac{dF}{dx}\right]_{x=L} &= 0 \end{aligned} \tag{140}$$

Will give our first Hermite cubic function,

$$H_1^e(x) = 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3} \tag{141}$$

We may obtain our Second Hermite Cubic Function by requiring our standard function to obey,

$$\begin{aligned} [F]_{x=0} &= 0 \\ \left[\frac{dF}{dx}\right]_{x=0} &= 1 \\ [F]_{x=L} &= 0 \\ \left[\frac{dF}{dx}\right]_{x=L} &= 0 \end{aligned} \tag{142}$$

Will give our second Hermite cubic function,

$$H_2^e(x) = x - \frac{2x^2}{L} + \frac{x^3}{L^2} \tag{143}$$

We may obtain our third Hermite Cubic Function by requiring our standard function to obey,

$$\begin{aligned} [F]_{x=0} &= 0 \\ \left[\frac{dF}{dx}\right]_{x=0} &= 0 \\ [F]_{x=L} &= 1 \\ \left[\frac{dF}{dx}\right]_{x=L} &= 0 \end{aligned} \tag{144}$$

Will give our third Hermite cubic function,

$$H_3^e(x) = \frac{3x^2}{L^2} - \frac{2x^3}{L^3} \tag{145}$$

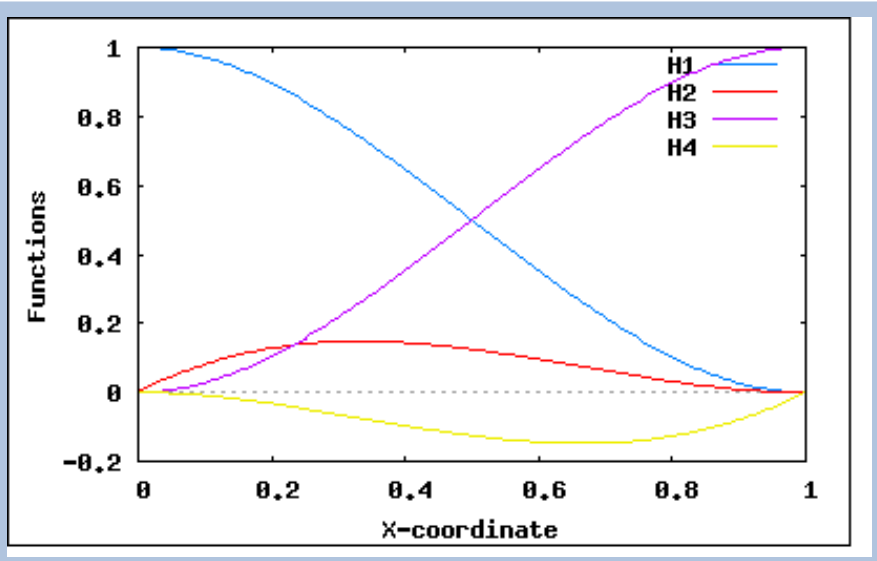
We may obtain our fourth Hermite Cubic Function by requiring our standard function to obey,

$$\begin{aligned} [F]_{x=0} &= 0 \\ \left[\frac{dF}{dx}\right]_{x=0} &= 0 \\ [F]_{x=L} &= 1 \\ \left[\frac{dF}{dx}\right]_{x=L} &= 0 \end{aligned} \tag{146}$$

Will give our fourth Hermite cubic function,

$$H_4^e(x) = \frac{x^3}{L^2} - \frac{x^2}{L} \tag{147}$$

These function, like before, may be plotted as,



6.1. 5^{th} order Hermite Functions

These third order functions are somewhat the culprits in weakening the finite element method. We propose a fifth order hermit functions. Which obey the following,

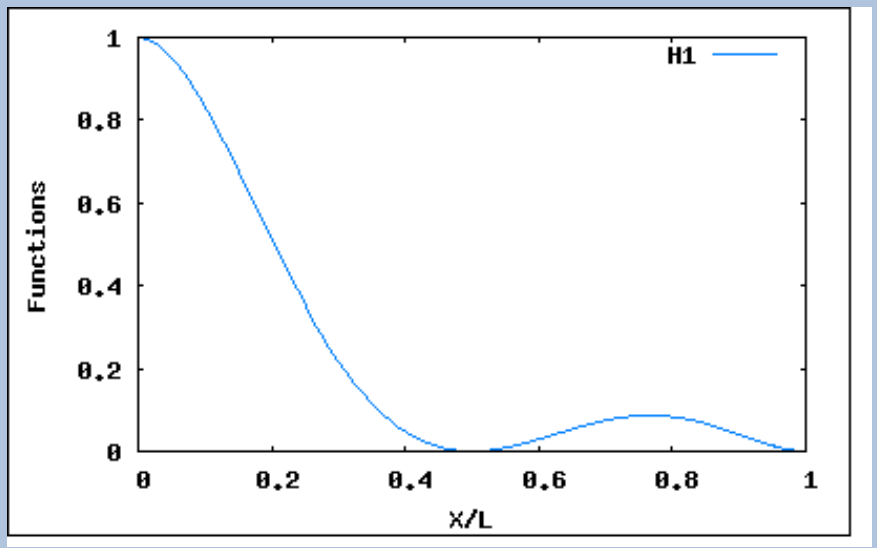
$$F(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 \tag{148}$$

By requiring this function to obey,

$$\begin{aligned}
[F]_{x=0} &= 1 \\
\left[\frac{dF}{dx}\right]_{x=0} &= 0 \\
[F]_{x=L/2} &= 0 \\
\left[\frac{dF}{dx}\right]_{x=L/2} &= 0 \\
[F]_{x=L} &= 0 \\
\left[\frac{dF}{dx}\right]_{x=L} &= 0
\end{aligned} \tag{149}$$

We obtain our first Hermite function as,

$$H_1^e(x) = 1 - \frac{22x^2}{L^2} + \frac{66x^3}{L^3} - \frac{68x^4}{L^4} + \frac{24x^5}{L^5} \tag{150}$$

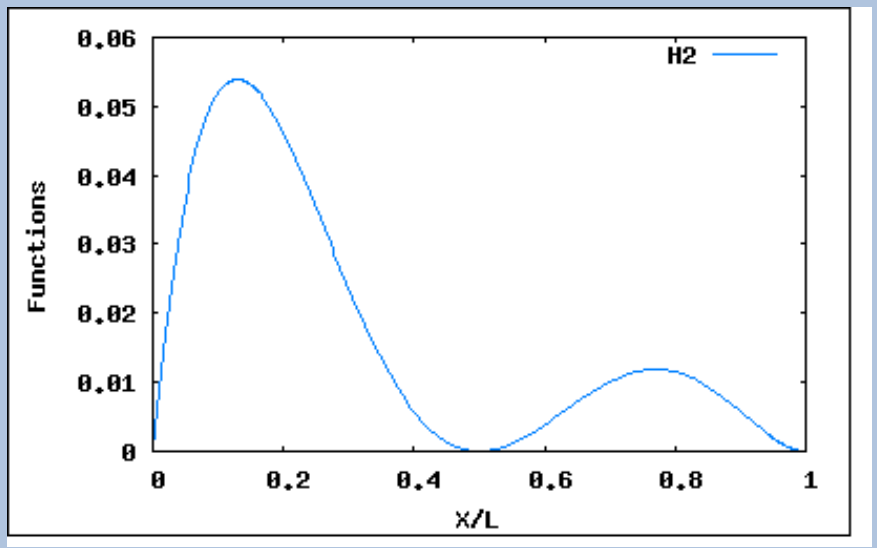


By requiring this function to obey,

$$\begin{aligned}
[F]_{x=0} &= 0 \\
\left[\frac{dF}{dx}\right]_{x=0} &= 1 \\
[F]_{x=L/2} &= 0 \\
\left[\frac{dF}{dx}\right]_{x=L/2} &= 0 \\
[F]_{x=L} &= 0 \\
\left[\frac{dF}{dx}\right]_{x=L} &= 0
\end{aligned} \tag{151}$$

We obtain our first Hermite function as,

$$H_2^e(x) = x - \frac{6x^2}{L} + \frac{13x^3}{L^2} - \frac{12x^4}{L^3} + \frac{4x^5}{L^4} \tag{152}$$

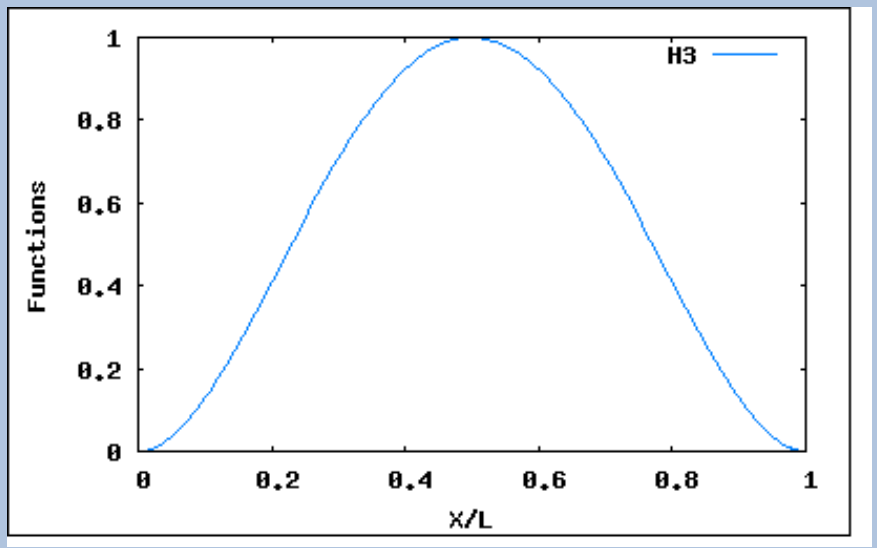


By requiring this function to obey,

$$\begin{aligned} [F]_{x=0} &= 0 \\ \left[\frac{dF}{dx}\right]_{x=0} &= 0 \\ [F]_{x=L/2} &= 1 \\ \left[\frac{dF}{dx}\right]_{x=L/2} &= 0 \\ [F]_{x=L} &= 0 \\ \left[\frac{dF}{dx}\right]_{x=L} &= 0 \end{aligned} \tag{153}$$

We obtain our first Hermite function as,

$$H_3^e(x) = \frac{16x^2}{L^2} - \frac{32x^3}{L^3} + \frac{16x^4}{L^4} \tag{154}$$

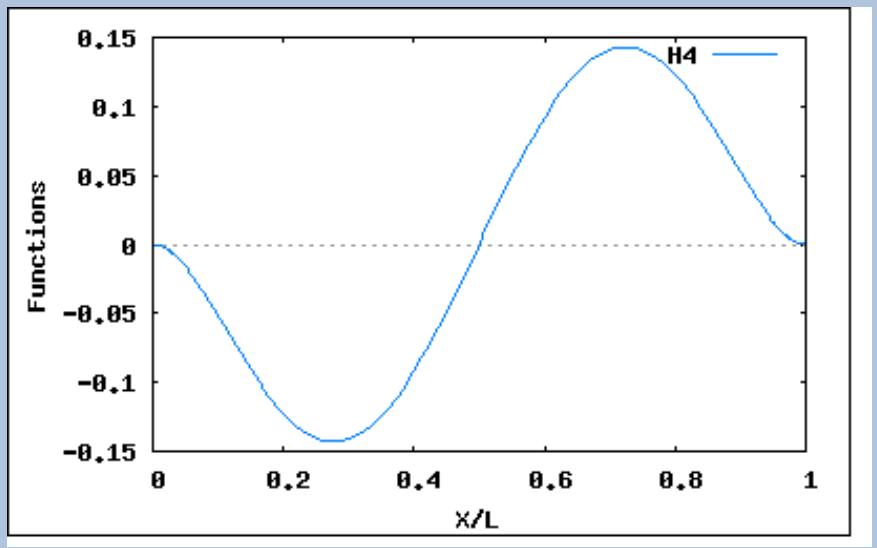


By requiring this function to obey,

$$\begin{aligned} [F]_{x=0} &= 0 \\ \left[\frac{dF}{dx}\right]_{x=0} &= 0 \\ [F]_{x=L/2} &= 0 \\ \left[\frac{dF}{dx}\right]_{x=L/2} &= 1 \\ [F]_{x=L} &= 0 \\ \left[\frac{dF}{dx}\right]_{x=L} &= 0 \end{aligned} \tag{155}$$

We obtain our first Hermite function as,

$$H_4^e(x) = -\frac{8x^2}{L} + \frac{32x^3}{L^2} - \frac{40x^4}{L^3} + \frac{16x^5}{L^4} \tag{156}$$

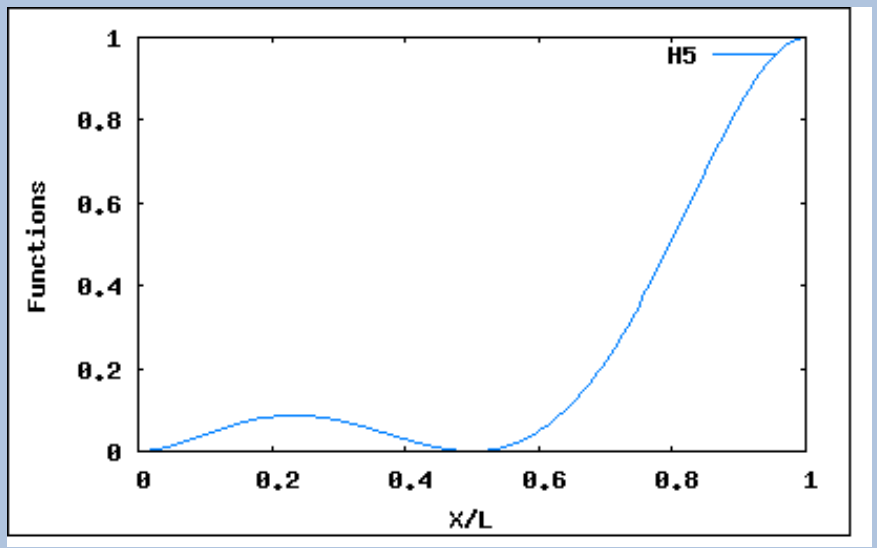


By requiring this function to obey,

$$\begin{aligned} [F]_{x=0} &= 0 \\ \left[\frac{dF}{dx}\right]_{x=0} &= 0 \\ [F]_{x=L/2} &= 0 \\ \left[\frac{dF}{dx}\right]_{x=L/2} &= 0 \\ [F]_{x=L} &= 1 \\ \left[\frac{dF}{dx}\right]_{x=L} &= 0 \end{aligned} \tag{157}$$

We obtain our first Hermite function as,

$$H_5^e(x) = \frac{7x^2}{L^2} - \frac{34x^3}{L^3} + \frac{52x^4}{L^4} - \frac{24x^5}{L^5} \tag{158}$$

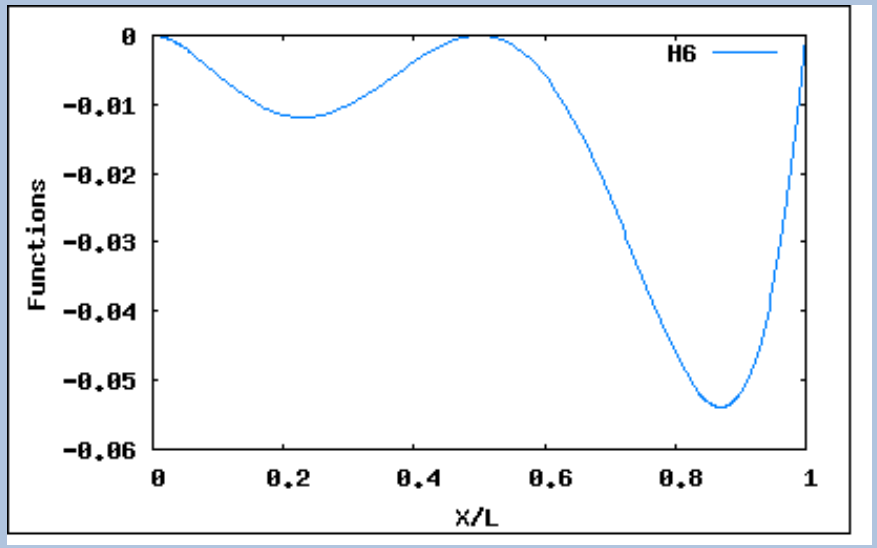


By requiring this function to obey,

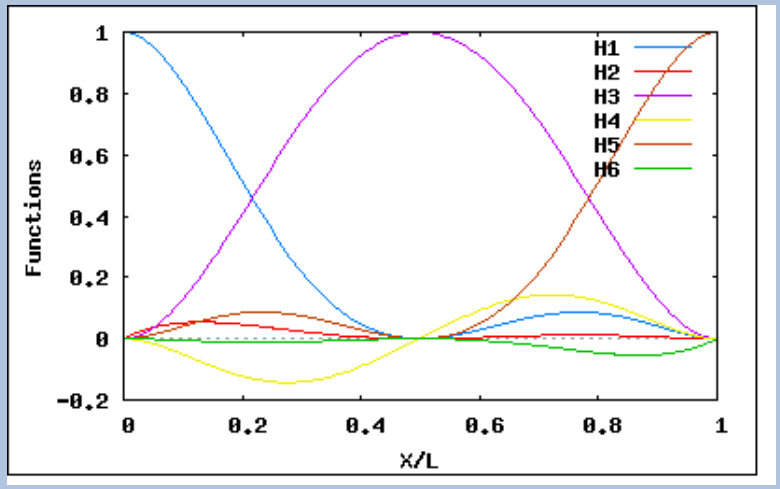
$$\begin{aligned} [F]_{x=0} &= 0 \\ \left[\frac{dF}{dx}\right]_{x=0} &= 0 \\ [F]_{x=L/2} &= 0 \\ \left[\frac{dF}{dx}\right]_{x=L/2} &= 0 \\ [F]_{x=L} &= 0 \\ \left[\frac{dF}{dx}\right]_{x=L} &= 1 \end{aligned} \tag{159}$$

We obtain our first Hermite function as,

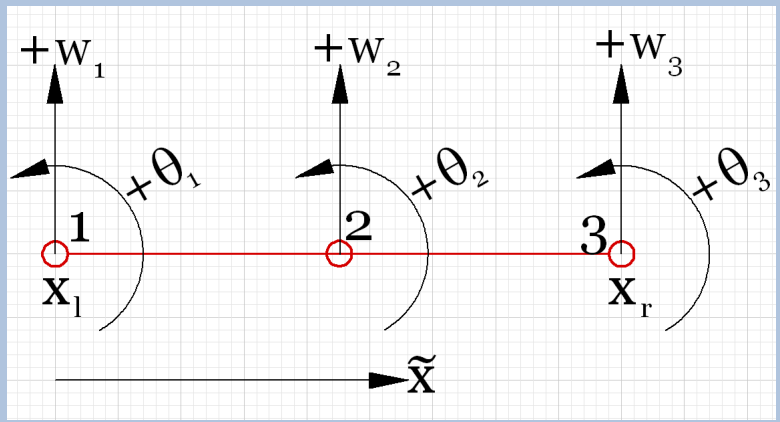
$$H_6^e(x) = -\frac{x^2}{L} + \frac{5x^3}{L^2} - \frac{8x^4}{L^3} + \frac{4x^5}{L^4} \tag{160}$$



All of these functions may be plotted together,



Our degrees of freedom are,

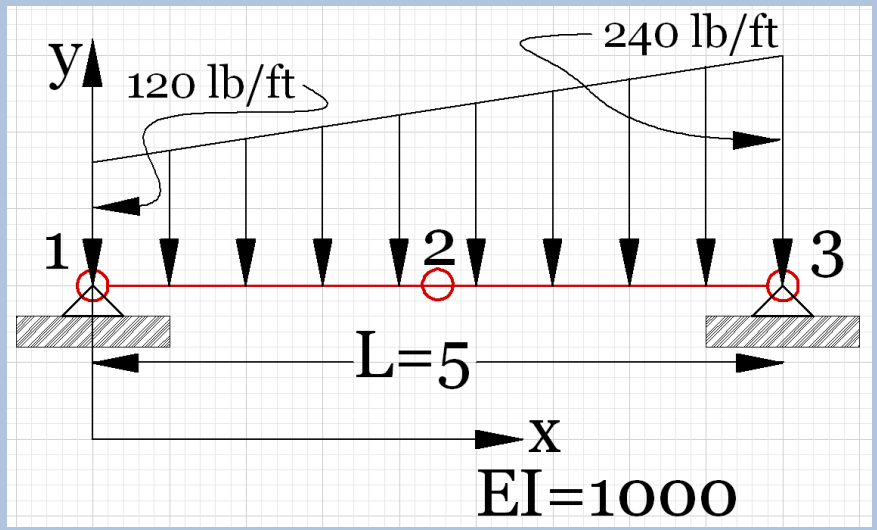


Our element equation becomes,

$$\frac{EI}{35L^3}\begin{bmatrix} 5092 & 1138L & -3584 & 1920L & -1508 & 242L \\ 1138L & 332L^2 & -896L & 320L^2 & -242L & 38L^2 \\ -3584 & -896L & 7168 & 0 & -3584 & 896L \\ 1920L & 320L^2 & 0 & 1280L^2 & -1920L & 320L^2 \\ -1508 & -242L & -3584 & -1920L & 5092 & -1138L \\ 242L & 38L^2 & 896L & 320L^2 & -1138L & 332L^2 \end{bmatrix}\begin{Bmatrix} W_1^e \\ \theta_1^e \\ W_2^e \\ \theta_2^e \\ W_3^e \\ \theta_3^e \end{Bmatrix}$$

$$= \frac{1}{420}\begin{Bmatrix} 19q_1L^2 + 98q_0L \\ 2q_1L^3 + 7q_0L^2 \\ 112q_1L^2 + 224q_0L \\ 8q_1L^3 \\ 79q_1L^2 + 98q_0L \\ -6q_1L^3 - 7q_0L^2 \end{Bmatrix} + \begin{Bmatrix} \mathcal{V}_1^e \\ \mathcal{M}_1^e \\ \mathcal{V}_2^e \\ \mathcal{M}_2^e \\ \mathcal{V}_3^e \\ \mathcal{M}_3^e \end{Bmatrix} \tag{161}$$

If we try to solve our simply supported, worst case to present, beam,



Our matrix becomes,

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1300.6 & 1897.1 & -1024.0 & 1828.6 & -76.57 & 217.14 \\ -819.20 & -1024.0 & 1638.4 & 0 & -819.20 & 1024.0 \\ 2194.3 & 1828.6 & 0 & 7314.3 & -2194.3 & 1828.6 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 276.57 & 217.14 & 1024.0 & 1828.6 & -1300.6 & 1897.1 \end{bmatrix} \begin{Bmatrix} W_1^e \\ \theta_1^e \\ W_2^e \\ \theta_2^e \\ W_3^e \\ \theta_3^e \end{Bmatrix} = \begin{Bmatrix} 0 \\ -64.28 \\ -480.00 \\ -57.14 \\ 0 \\ 85.71 \end{Bmatrix} + \begin{Bmatrix} 0 \\ \mathcal{M}_1^e \\ \mathcal{V}_2^e \\ \mathcal{M}_2^e \\ 0 \\ \mathcal{M}_3^e \end{Bmatrix} \tag{162}$$

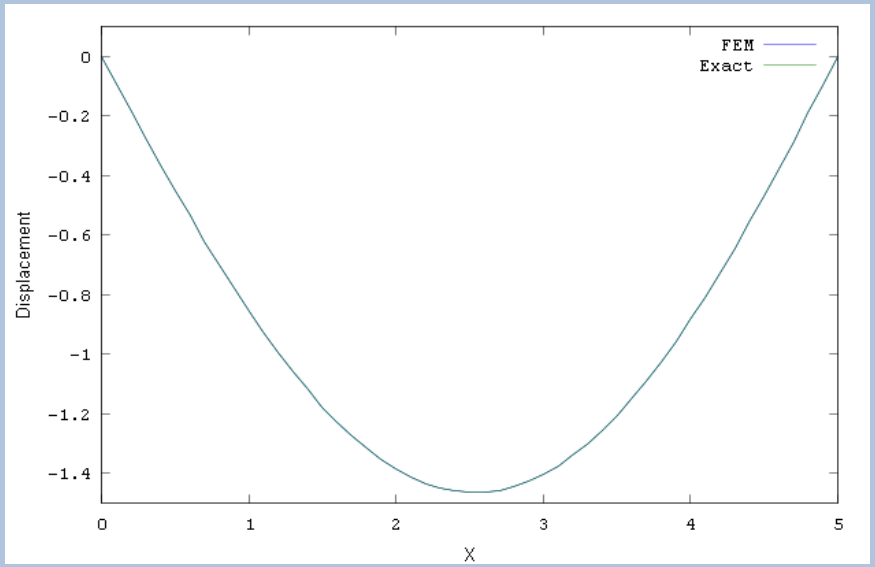
Since there are no applied point loads or moments,

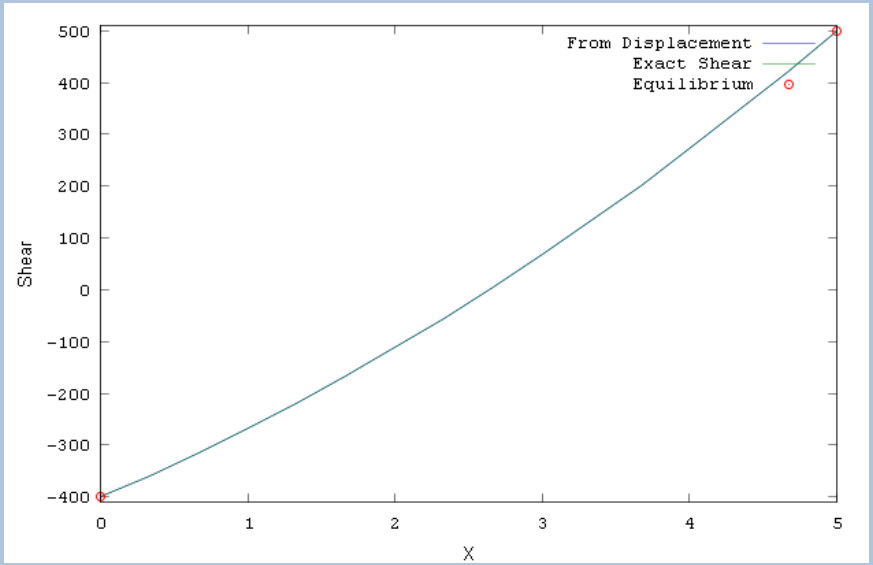
$$\begin{Bmatrix} 0 \\ \mathcal{M}_1^e \\ \mathcal{V}_2^e \\ \mathcal{M}_2^e \\ 0 \\ \mathcal{M}_3^e \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{163}$$

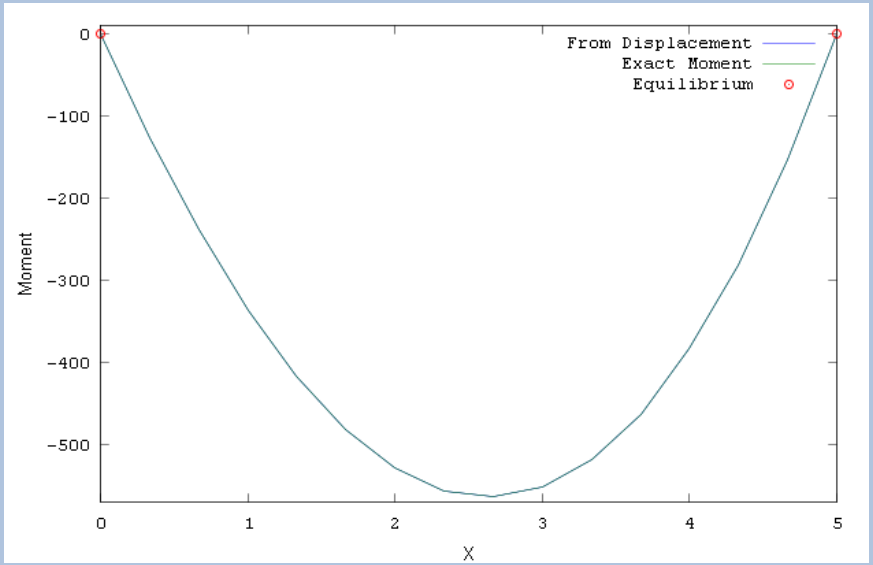
We obtain a solution of,

$$\begin{Bmatrix} W_1^e \\ \theta_1^e \\ W_2^e \\ \theta_2^e \\ W_3^e \\ \theta_3^e \end{Bmatrix} = \begin{Bmatrix} 0 \\ -0.91667 \\ -1.46484 \\ -0.01823 \\ 0 \\ 0.95833 \end{Bmatrix} \tag{164}$$

Plots of our solution reveal something interesting,


FEM
Exact
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-1.4
Displacement
0
1
2
3
4
5
X


From Displacement
Exact Shear
Equilibrium
Shear
X
500
400
300
200
100
0
-100
-200
-300
-400
0
1
2
3
4
5

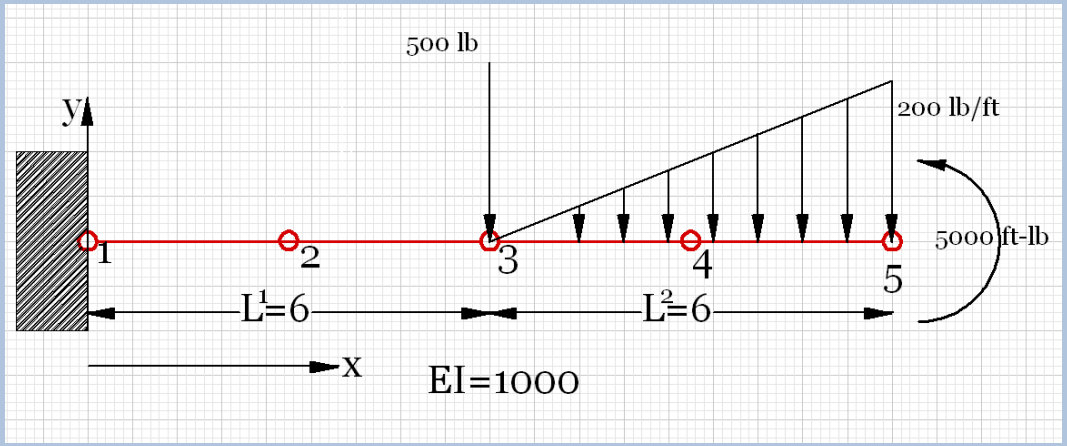


THE FINITE ELEMENT METHOD OBTAINED THE EXACT SOLUTION FOR THE DEFLECTION, SHEAR AND MOMENT!

For this reason, we shall use ONLY the fifth order Hermite basis function! This fifth order Hermite basis function will obtain the exact solution for beams with, up to, linearly distributed loads. We will now investigate several Multi-Element examples.

6.2. Multi Element Case 5

We begin with the following beam,



There are 2 element, each of length 6. For element 1 we have,

$$\begin{bmatrix} 673.5 & 903.2 & -474.1 & 1523.8 & -199.5 & 192.1 & 0.0 & 0.0 & 0.0 & 0.0 \\ 903.2 & 1581.0 & -711.1 & 1523.8 & -192.1 & 181.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ -474.1 & -711.1 & 948.1 & 0.0 & -474.1 & 711.1 & 0.0 & 0.0 & 0.0 & 0.0 \\ 1523.8 & 1523.8 & 0.0 & 6095.2 & -1523.8 & 1523.8 & 0.0 & 0.0 & 0.0 & 0.0 \\ -199.5 & -192.1 & -474.1 & -1523.8 & 673.5 & -903.2 & 0.0 & 0.0 & 0.0 & 0.0 \\ 192.1 & 181.0 & 711.1 & 1523.8 & -903.2 & 1581.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix} \begin{Bmatrix} W_1 \\ \theta_1 \\ W_2 \\ \theta_2 \\ W_3 \\ \theta_3 \\ W_4 \\ \theta_4 \\ W_5 \\ \theta_5 \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} \mathcal{V}_1^1 \\ \mathcal{M}_1^1 \\ \mathcal{V}_2^1 \\ \mathcal{M}_2^1 \\ \mathcal{V}_3^1 \\ \mathcal{M}_3^1 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{165}$$

For element 2 we have,

$$\begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 673.5 & 903.2 & -474.1 & 1523.8 & -199.5 & 192.1 \\ 0.0 & 0.0 & 0.0 & 0.0 & 903.2 & 1581.0 & -711.1 & 1523.8 & -192.1 & 181.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & -474.1 & -711.1 & 948.1 & 0.0 & -474.1 & 711.1 \\ 0.0 & 0.0 & 0.0 & 0.0 & 1523.8 & 1523.8 & 0.0 & 6095.2 & -1523.8 & 1523.8 \\ 0.0 & 0.0 & 0.0 & 0.0 & -199.5 & -192.1 & -474.1 & -1523.8 & 673.5 & -903.2 \\ 0.0 & 0.0 & 0.0 & 0.0 & 192.1 & 181.0 & 711.1 & 1523.8 & -903.2 & 1581.0 \end{bmatrix} \begin{Bmatrix} W_1 \\ \theta_1 \\ W_2 \\ \theta_2 \\ W_3 \\ \theta_3 \\ W_4 \\ \theta_4 \\ W_5 \\ \theta_5 \end{Bmatrix}$$

$$= \begin{Bmatrix} 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \\ -54.3 \\ -34.3 \\ -320.0 \\ -137.1 \\ -225.7 \\ 85.7 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \mathcal{V}_1^2 \\ \mathcal{M}_1^2 \\ \mathcal{V}_2^2 \\ \mathcal{M}_2^2 \\ \mathcal{V}_3^2 \\ \mathcal{M}_3^2 \end{Bmatrix} \tag{166}$$

Adding these 2 equation together we have,

$$\begin{bmatrix} 673.5 & 903.2 & -474.1 & 1523.8 & -199.5 & 192.1 & 0.0 & 0.0 & 0.0 & 0.0 \\ 903.2 & 1581.0 & -711.1 & 1523.8 & -192.1 & 181.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ -474.1 & -711.1 & 948.1 & 0.0 & -474.1 & 711.1 & 0.0 & 0.0 & 0.0 & 0.0 \\ 1523.8 & 1523.8 & 0.0 & 6095.2 & -1523.8 & 1523.8 & 0.0 & 0.0 & 0.0 & 0.0 \\ -199.5 & -192.1 & -474.1 & -1523.8 & 1347.1 & 0.0 & -474.1 & 1523.8 & -199.5 & 192.1 \\ 192.1 & 181.0 & 711.1 & 1523.8 & 0.0 & 3161.9 & -711.1 & 1523.8 & -192.1 & 181.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & -474.1 & -711.1 & 948.1 & 0.0 & -474.1 & 711.1 \\ 0.0 & 0.0 & 0.0 & 0.0 & 1523.8 & 1523.8 & 0.0 & 6095.2 & -1523.8 & 1523.8 \\ 0.0 & 0.0 & 0.0 & 0.0 & -199.5 & -192.1 & -474.1 & -1523.8 & 673.5 & -903.2 \\ 0.0 & 0.0 & 0.0 & 0.0 & 192.1 & 181.0 & 711.1 & 1523.8 & -903.2 & 1581.0 \end{bmatrix} \begin{Bmatrix} W_1 \\ \theta_1 \\ W_2 \\ \theta_2 \\ W_3 \\ \theta_3 \\ W_4 \\ \theta_4 \\ W_5 \\ \theta_5 \end{Bmatrix}$$

$$= \begin{Bmatrix} 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \\ -54.3 \\ -34.3 \\ -320.0 \\ -137.1 \\ -225.7 \\ 85.7 \end{Bmatrix} + \begin{Bmatrix} \mathcal{V}_1^1 \\ \mathcal{M}_1^1 \\ \mathcal{V}_2^1 \\ \mathcal{M}_2^1 \\ \mathcal{V}_3^1 + \mathcal{V}_1^2 \\ \mathcal{M}_3^1 + \mathcal{M}_1^2 \\ \mathcal{V}_2^2 \\ \mathcal{M}_2^2 \\ \mathcal{V}_3^2 \\ \mathcal{M}_3^2 \end{Bmatrix} \tag{167}$$

We first knock out the essential boundary conditions, $W_1 = \theta_1 = 0$ to obtain,

Adding these 2 equation together we have,

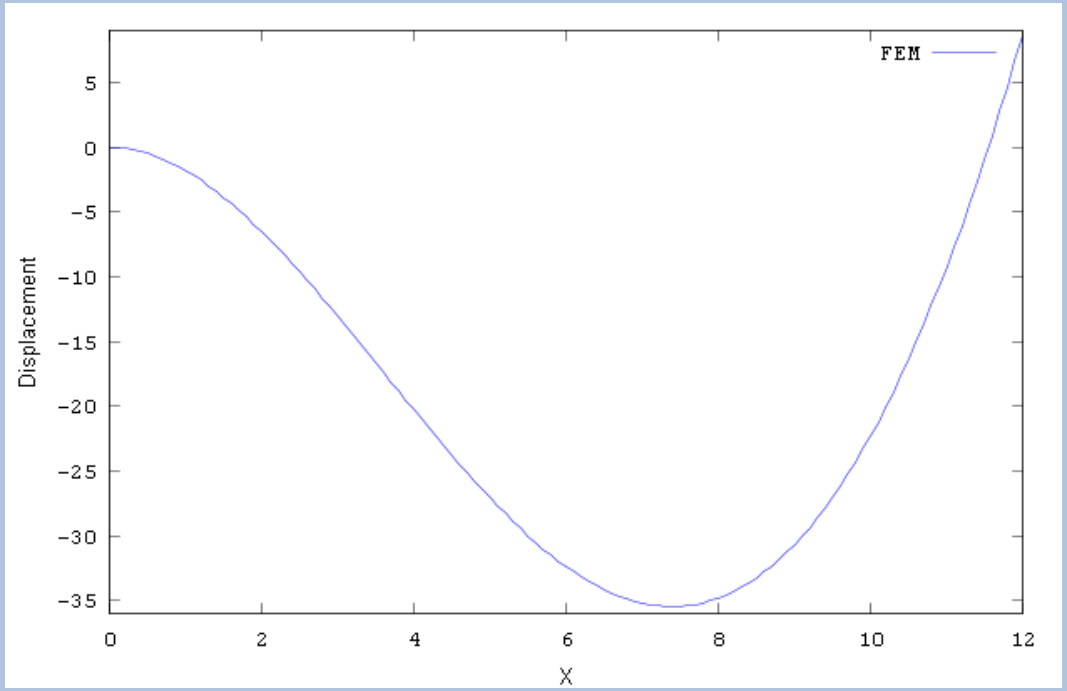
$$\begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ -474.1 & -711.1 & 948.1 & 0.0 & -474.1 & 711.1 & 0.0 & 0.0 & 0.0 & 0.0 \\ 1523.8 & 1523.8 & 0.0 & 6095.2 & -1523.8 & 1523.8 & 0.0 & 0.0 & 0.0 & 0.0 \\ -199.5 & -192.1 & -474.1 & -1523.8 & 1347.1 & 0.0 & -474.1 & 1523.8 & -199.5 & 192.1 \\ 192.1 & 181.0 & 711.1 & 1523.8 & 0.0 & 3161.9 & -711.1 & 1523.8 & -192.1 & 181.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & -474.1 & -711.1 & 948.1 & 0.0 & -474.1 & 711.1 \\ 0.0 & 0.0 & 0.0 & 0.0 & 1523.8 & 1523.8 & 0.0 & 6095.2 & -1523.8 & 1523.8 \\ 0.0 & 0.0 & 0.0 & 0.0 & -199.5 & -192.1 & -474.1 & -1523.8 & 673.5 & -903.2 \\ 0.0 & 0.0 & 0.0 & 0.0 & 192.1 & 181.0 & 711.1 & 1523.8 & -903.2 & 1581.0 \end{bmatrix} \begin{Bmatrix} W_1 \\ \theta_1 \\ W_2 \\ \theta_2 \\ W_3 \\ \theta_3 \\ W_4 \\ \theta_4 \\ W_5 \\ \theta_5 \end{Bmatrix}$$

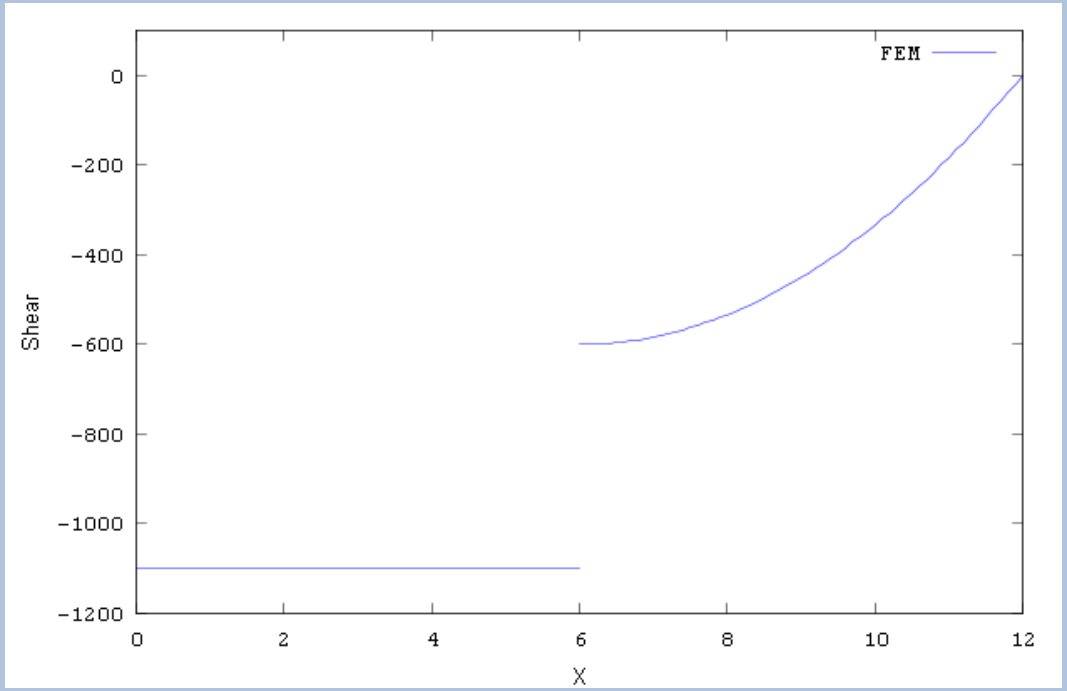
$$= \begin{Bmatrix} 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \\ -54.3 \\ -34.3 \\ -320.0 \\ -137.1 \\ -225.7 \\ 85.7 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ \mathcal{V}_2^1 \\ \mathcal{M}_2^1 \\ \mathcal{V}_3^1 + \mathcal{V}_1^2 \\ \mathcal{M}_3^1 + \mathcal{M}_1^2 \\ \mathcal{V}_2^2 \\ \mathcal{M}_2^2 \\ \mathcal{V}_3^2 \\ \mathcal{M}_3^2 \end{Bmatrix} \tag{168}$$

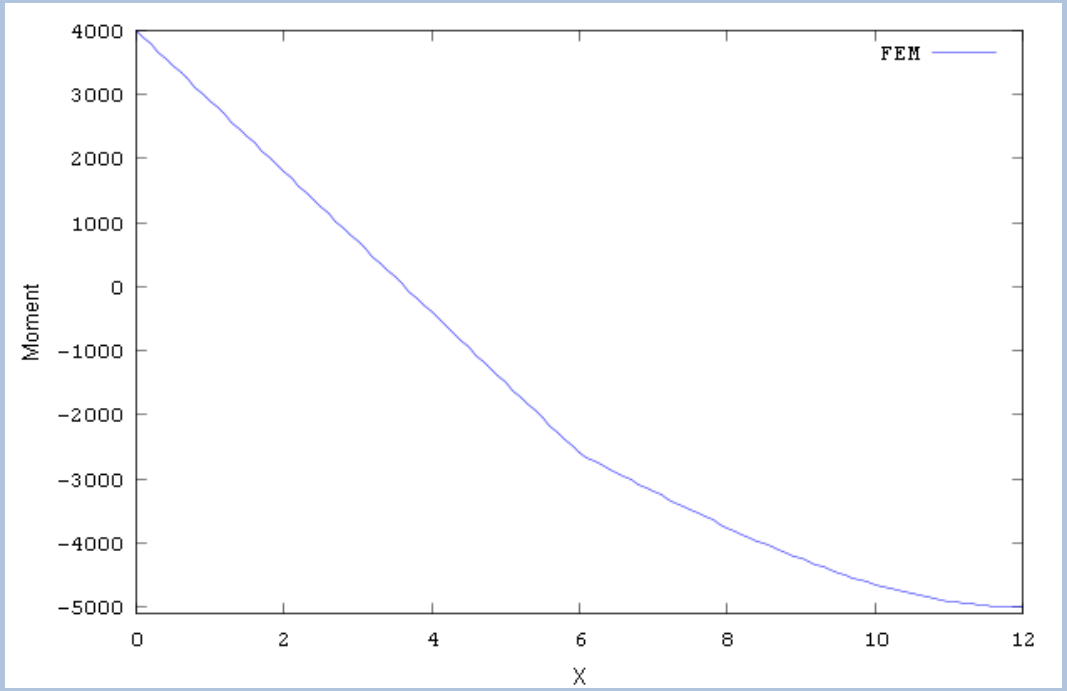
Next we recognize that there is two applied loads, $\mathcal{V}_3^1 + \mathcal{V}_1^2 = -500$ and $\mathcal{M}_3^2 = 5000$

$$\begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ -474.1 & -711.1 & 948.1 & 0.0 & -474.1 & 711.1 & 0.0 & 0.0 & 0.0 & 0.0 \\ 1523.8 & 1523.8 & 0.0 & 6095.2 & -1523.8 & 1523.8 & 0.0 & 0.0 & 0.0 & 0.0 \\ -199.5 & -192.1 & -474.1 & -1523.8 & 1347.1 & 0.0 & -474.1 & 1523.8 & -199.5 & 192.1 \\ 192.1 & 181.0 & 711.1 & 1523.8 & 0.0 & 3161.9 & -711.1 & 1523.8 & -192.1 & 181.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & -474.1 & -711.1 & 948.1 & 0.0 & -474.1 & 711.1 \\ 0.0 & 0.0 & 0.0 & 0.0 & 1523.8 & 1523.8 & 0.0 & 6095.2 & -1523.8 & 1523.8 \\ 0.0 & 0.0 & 0.0 & 0.0 & -199.5 & -192.1 & -474.1 & -1523.8 & 673.5 & -903.2 \\ 0.0 & 0.0 & 0.0 & 0.0 & 192.1 & 181.0 & 711.1 & 1523.8 & -903.2 & 1581.0 \end{bmatrix} \begin{Bmatrix} W_1 \\ \theta_1 \\ W_2 \\ \theta_2 \\ W_3 \\ \theta_3 \\ W_4 \\ \theta_4 \\ W_5 \\ \theta_5 \end{Bmatrix}$$

$$= \begin{Bmatrix} 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \\ -54.3 \\ -34.3 \\ -320.0 \\ -137.1 \\ -225.7 \\ 85.7 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -500 \\ 0 \\ 0 \\ 0 \\ 0 \\ 5000 \end{Bmatrix} \tag{169}$$

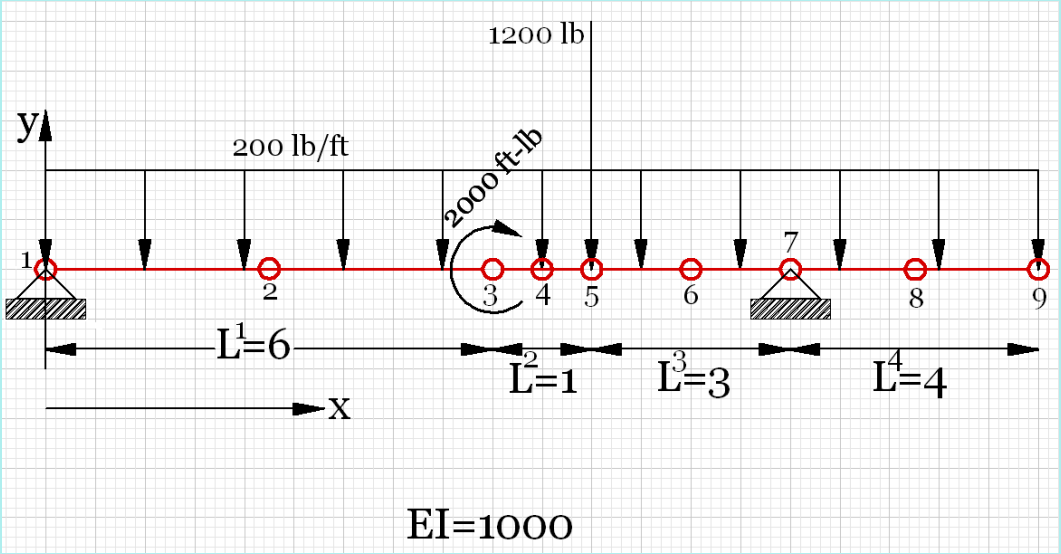

FEM
Displacement
X
5
0
-5
-10
-15
-20
-25
-30
-35
0
2
4
6
8
10
12


FEM
Shear
X
0
-200
-400
-600
-800
-1000
-1200
0
2
4
6
8
10
12



These solutions are exact.

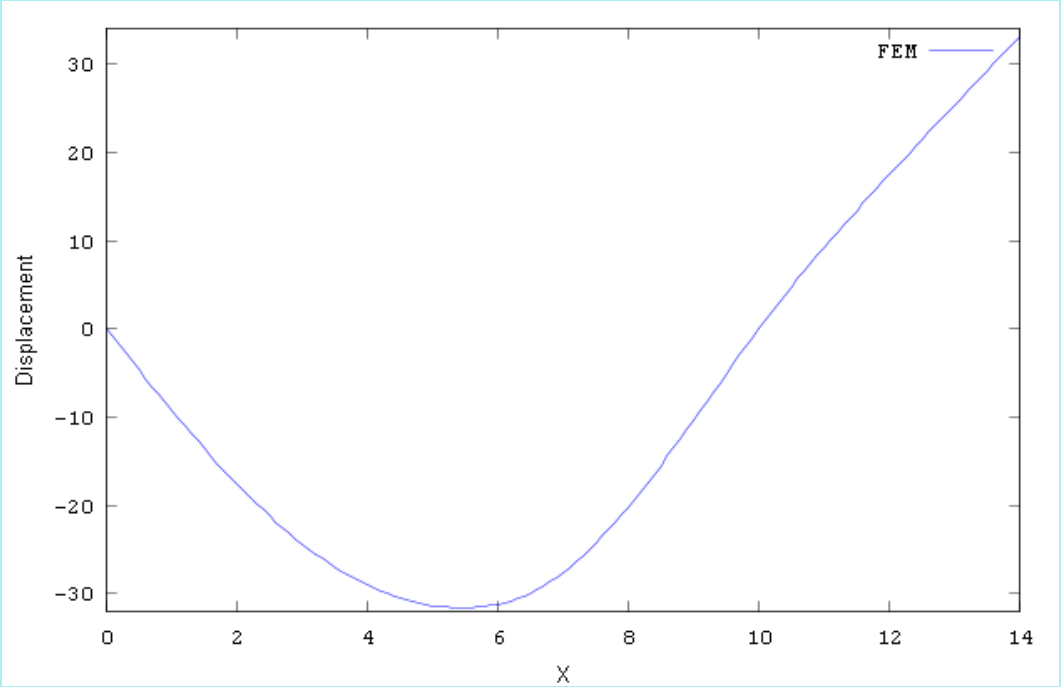
7. Multi Element Beam Case 6



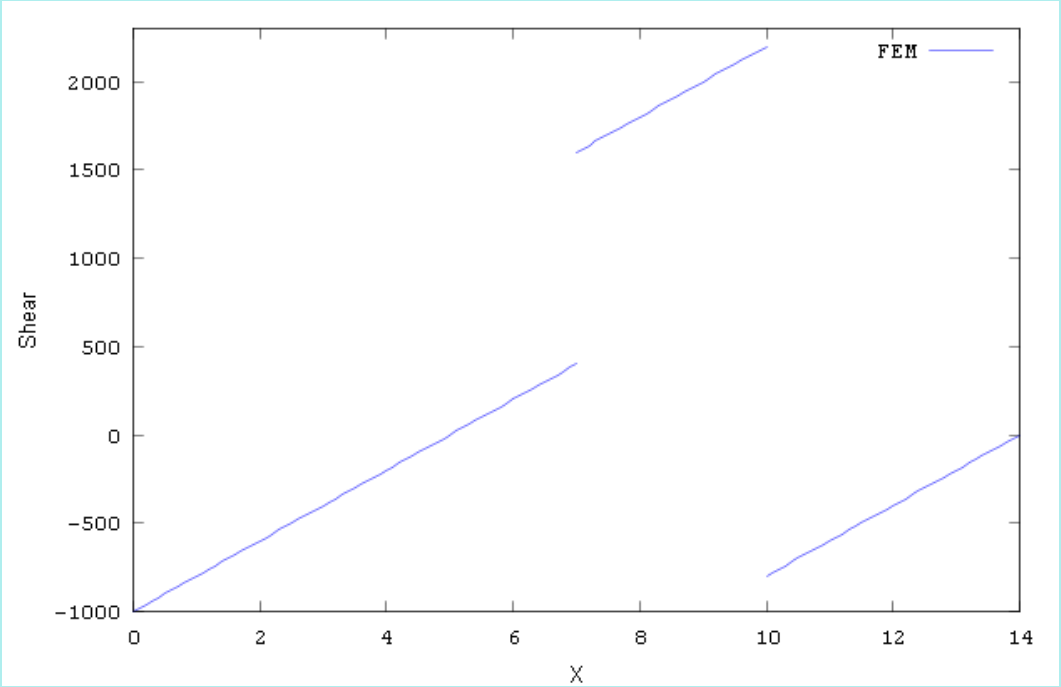
From this figure we see the following boundary conditions,

- $W_1 = 0$
- $W_7 = 0$
- $\mathcal{M}_3 = -2000$
- $\mathcal{V}_5 = -1200$

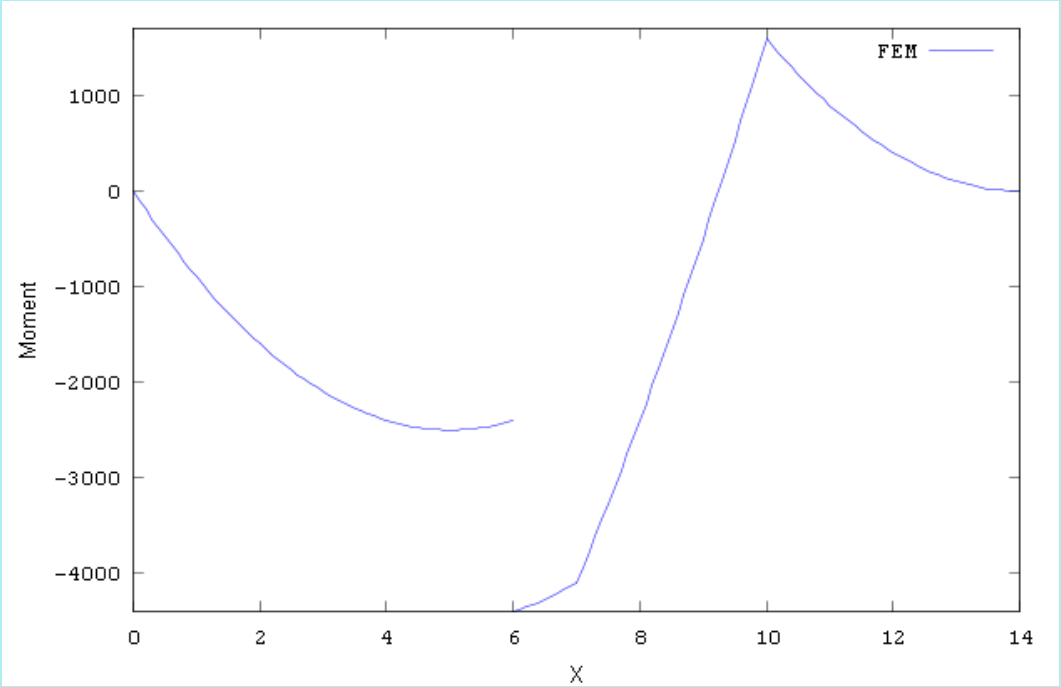
In addition the distributed loads are $q_1 = 0$ and $q_0 = -200$. Assembling these matrices, See Octave file, our solution becomes;



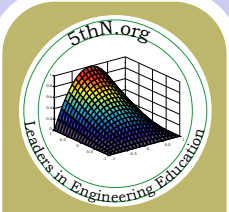
FEM and Exact Solution are the Same!



FEM and Exact Solution are the Same!



FEM and Exact Solution are the Same!



THE END

FEM of Beams and Bars

- Next Course Deals Beams-Bars oriented in plane trusses!
- Thank You for Training With US

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