

Plane Truss Analysis using Finite Elements Methods

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Abstract

In this course we will cover the theory behind the development of finite elements for plane truss systems. We will,

- Reviewing the development of the FEM equations for bars and beams.
- Develop general coordinate transformations.
- Use these coordinate transformation to rotate any beam-bar in a 2-D plane.
- Cover the assembly process of our beam-bar system.
- Cover software implementation of the system.
- Cover beam-bar geometry input into DesignCAD for a plane truss system.
- Cover input of material properties in DesignCAD.
- Cover boundary conditions input in DesignCAD.
- Review the simulation process.
- Cover post processing the data using Paraview.
- Examine Displacement, Shear and Moments in Paraview.

http://5thn.org/

1. Introductions

We will begin with the standard FEM simulations of beam-bar. This include a third order Hermite cubic representation of beam displacement and linear displacement of bars. However, we must realize that these displacements may not be accurate enough for including distributed loads. To this end, we will move into a higher order representation of our displacements.

1.1. Bar Equation

Our differential equation for bars under a linearly varying distributed load is,

$$\frac{d}{dx}\left[EA\frac{dU}{dx}\right] = a_0 + a_1 x \tag{1}$$

Using our linear basis/shape functions,

$$\begin{aligned} \psi_1 &= \left(1 - \frac{x}{L}\right) \\ \psi_2 &= \frac{x}{L} \end{aligned} \tag{2}$$

Our finite element matrix equation for bars is, (linear basis function therefore linear displacements)

$$\frac{AE}{L^e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} U_1^e \\ U_2^e \end{Bmatrix} = a_0\begin{Bmatrix} \frac{L}{2} \\ \frac{L}{2} \end{Bmatrix} + a_1\begin{Bmatrix} \frac{L^2}{6} \\ \frac{L^2}{3} \end{Bmatrix} + \begin{Bmatrix} \mathcal{P}_1^e \\ \mathcal{P}_2^e \end{Bmatrix} \tag{3}$$

1.2. Beam Equations

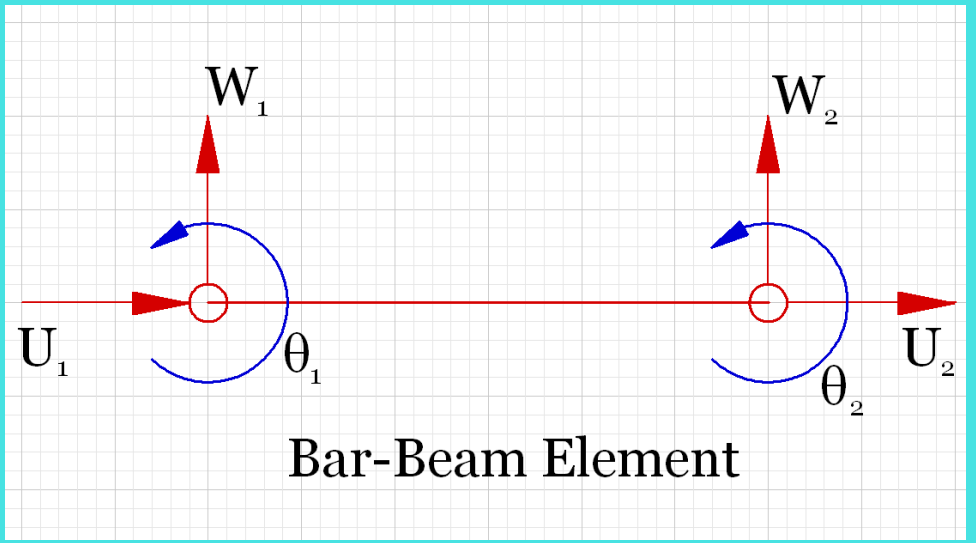
Our differential equation for bars under a linearly varying distributed load is,

$$\frac{d^2}{dx^2}\left[EI\frac{d^2W(x)}{dx^2}\right] = q_0 + q_1 x \tag{4}$$

Our finite element matrix equation for beams is, (Hermite cubic basis function therefore cubic displacements)

$$EI\begin{bmatrix} \frac{12}{L^3} & \frac{6}{L^2} & -\frac{12}{L^3} & \frac{6}{L^2} \\ \frac{6}{L^2} & \frac{4}{L} & -\frac{6}{L^2} & \frac{2}{L} \\ -\frac{12}{L^3} & -\frac{6}{L^2} & \frac{12}{L^3} & -\frac{6}{L^2} \\ \frac{6}{L^2} & \frac{2}{L} & -\frac{6}{L^2} & \frac{4}{L} \end{bmatrix}\begin{Bmatrix} W_1^e \\ \theta_1^e \\ W_2^e \\ \theta_2^e \end{Bmatrix} = q_1\begin{Bmatrix} \frac{3L^2}{20} \\ \frac{L^3}{30} \\ \frac{7L^2}{20} \\ -\frac{L^3}{20} \end{Bmatrix} + q_0\begin{Bmatrix} \frac{L}{2} \\ \frac{L^2}{12} \\ \frac{L}{2} \\ -\frac{L^2}{12} \end{Bmatrix} + \begin{Bmatrix} \mathcal{V}_l^e \\ \mathcal{M}_l^e \\ \mathcal{V}_r^e \\ \mathcal{M}_r^e \end{Bmatrix} \tag{5}$$

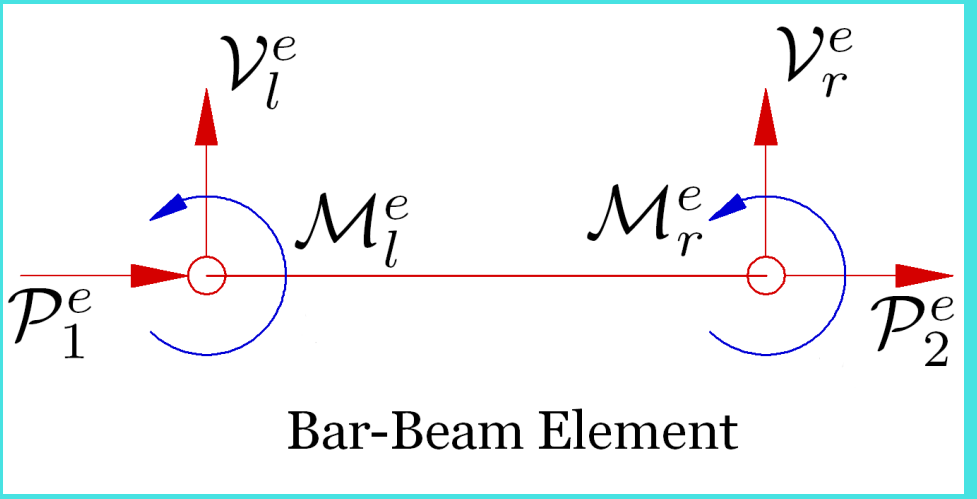
Our sign convention is shown below,



We must decide how we are going to combine our elements together. To this end we will use a standard method of considering the displacement of the bar first, then the displacement of the beam and finally the rotation of the beam. This will give us the following matrix,

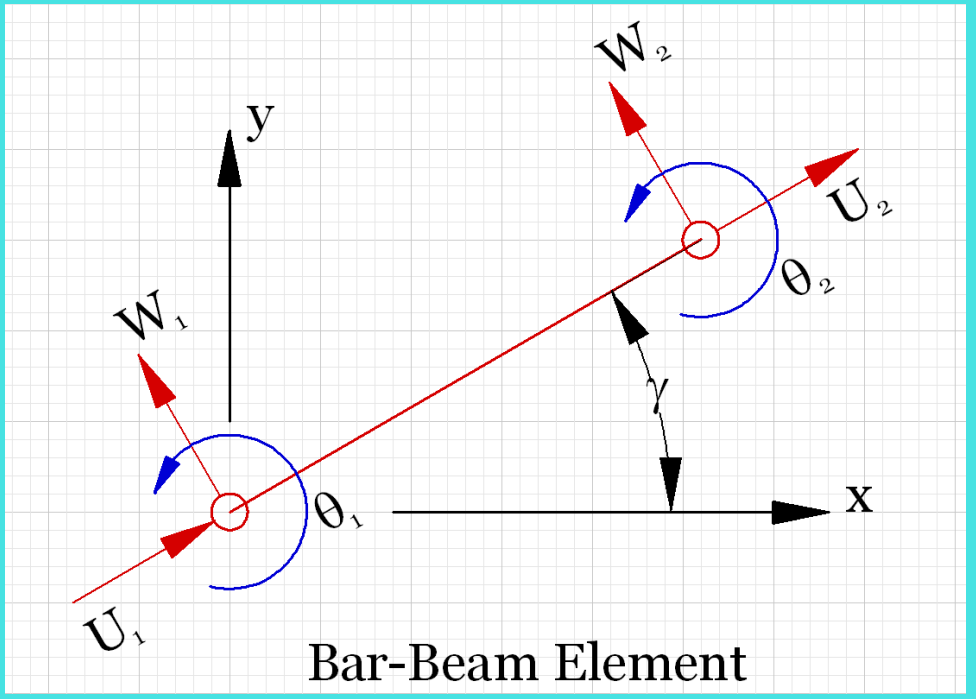
$$\begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}\begin{Bmatrix} U_1^e \\ W_1^e \\ \theta_1^e \\ U_2^e \\ W_2^e \\ \theta_2^e \end{Bmatrix} = \begin{Bmatrix} \frac{a_0 L}{2} + \frac{a_1 L^2}{6} \\ \frac{q_1 3L^2}{20} + \frac{q_0 L}{2} \\ \frac{q_1 L^3}{30} + \frac{q_0 L^2}{12} \\ \frac{a_0 L}{2} + \frac{a_1 L^2}{3} \\ \frac{q_1 7L^2}{20} + \frac{q_0 L}{2} \\ -\frac{q_1 L^3}{20} - \frac{q_0 L^2}{12} \end{Bmatrix} + \begin{Bmatrix} \mathcal{P}_1^e \\ \mathcal{V}_l^e \\ \mathcal{M}_l^e \\ \mathcal{P}_2^e \\ \mathcal{V}_r^e \\ \mathcal{M}_r^e \end{Bmatrix} \tag{6}$$

Our Right hand side force vector has the following notation,



1.3. Bar-Beam Orientation/Rotation

We now consider the possibility that our bar-beam is oriented in a plane at an arbitrary angle.

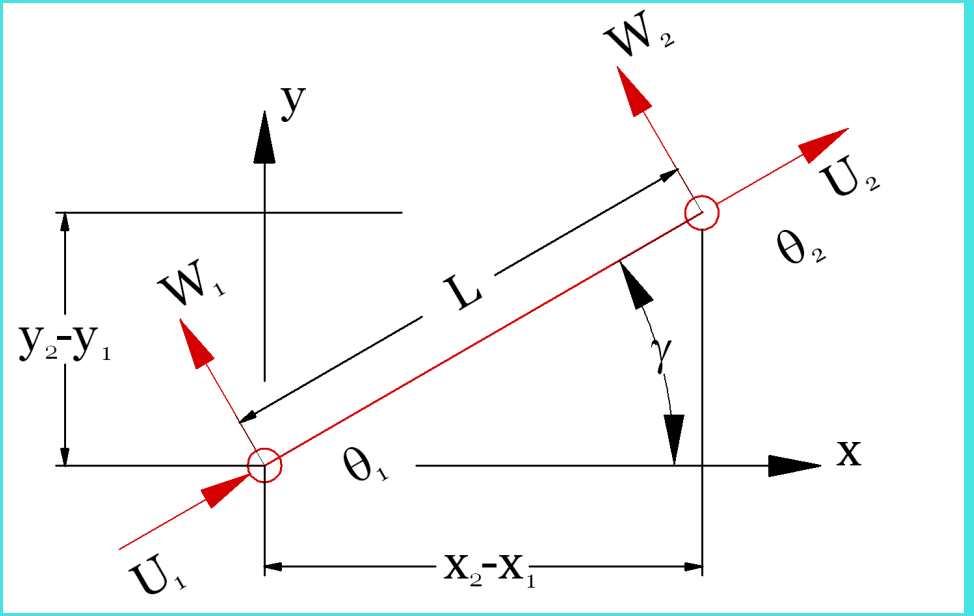


We will now rotate our displacement vectors into the x-y coordinates. To this end we will define our orientation by both our original coordinates and a new system oriented with the beam. Our angle γ may be easily determined by using the relationship,

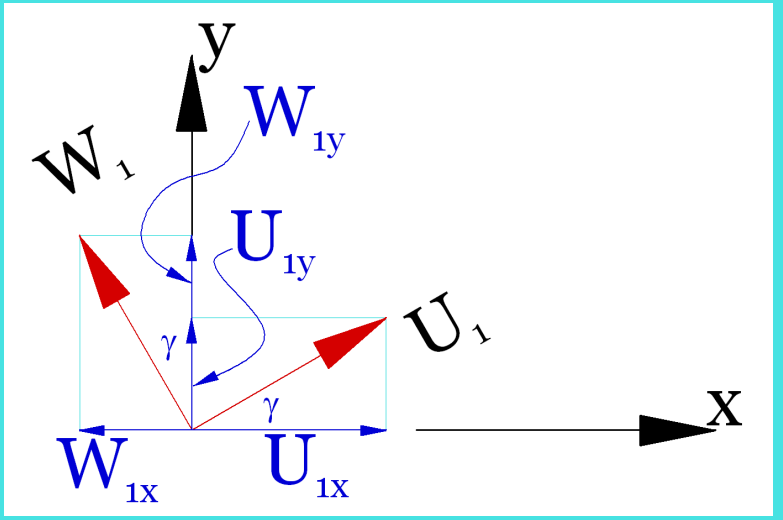
$$\sin(\gamma) = \frac{y_2 - y_1}{L} \tag{7}$$

and

$$\cos(\gamma) = \frac{x_2 - x_1}{L} \tag{8}$$



We may now consider the deflection of any node by using a coordinate transformation,



We first transform the bar deflection U to our $x-y$ coordinates.

$$\vec{U}_1 = \vec{U}_{1x}\hat{\imath} + \vec{U}_{1y}\hat{\jmath} = \|\vec{U}\|\cos(\gamma)\,\hat{\imath} + \|\vec{U}\|\sin(\gamma)\,\hat{\jmath} \tag{9}$$

We then transform our beam deflection W as

$$\vec{W}_1 = \vec{W}_{1x}\hat{\imath} + \vec{W}_{1y}\hat{\jmath} = -\|\vec{W}\|\sin(\gamma)\,\hat{\imath} + \|\vec{W}\|\cos(\gamma)\,\hat{\jmath} \tag{10}$$

If we want our deflection in standard $x-y$ coordinates we can define these deflections as, $\mathcal{W}$ is the deflection in the y direction and $\mathcal{U}$ is the deflection in the x direction. Carefully observing our displacements we have,

$$\mathcal{U}_1 = U_1\cos(\gamma) - W_1\sin(\gamma) \tag{11}$$

$$\mathcal{W}_1 = U_1\sin(\gamma) + W_1\cos(\gamma) \tag{12}$$

Where,

- $U_1 = \|\vec{U}\|$
- $W_1 = \|\vec{W}\|$

These equations may be written as,

$$\left\{ \begin{array}{c} \mathcal{U}_1 \\ \mathcal{W}_1 \end{array} \right\} = \left[\begin{array}{cc} \cos(\gamma) & -\sin(\gamma) \\ \sin(\gamma) & \cos(\gamma) \end{array} \right] \left\{ \begin{array}{c} U_1 \\ W_1 \end{array} \right\} \tag{13}$$

This equation is the transformation matrix from Beam coordinates (U_1 and W_1) to physical displacement coordinates, ($\mathcal{U}_1$ and $\mathcal{W}_1$). To make this transformation complete we must also include our rotation at the beam ends. Since we are rotating our elements in the plane this transformation matrix is,

$$\left\{ \begin{array}{c} \mathcal{U}_1 \\ \mathcal{W}_1 \\ \theta_1 \end{array} \right\} = \left[\begin{array}{ccc} \cos(\gamma) & -\sin(\gamma) & 0 \\ \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{array} \right] \left\{ \begin{array}{c} U_1 \\ W_1 \\ \theta_1 \end{array} \right\} \tag{14}$$

We must transform both ends of our beam-bar element. Therefore the transformation matrix becomes,

$$\left\{ \begin{array}{c} \mathcal{U}_1 \\ \mathcal{W}_1 \\ \theta_1 \\ \mathcal{U}_2 \\ \mathcal{W}_2 \\ \theta_2 \end{array} \right\} = \left[\begin{array}{cccccc} \cos(\gamma) & -\sin(\gamma) & 0 & 0 & 0 & 0 \\ \sin(\gamma) & \cos(\gamma) & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos(\gamma) & -\sin(\gamma) & 0 \\ 0 & 0 & 0 & \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right] \left\{ \begin{array}{c} U_1 \\ W_1 \\ \theta_1 \\ U_2 \\ W_2 \\ \theta_2 \end{array} \right\} \tag{15}$$

In symbolic form we have,

$$\{\mathcal{D}\} = [T]\{D\} \tag{16}$$

We may also verify these following, see Maxima and Octave code.

$$\left\{ \begin{array}{c} U_1 \\ W_1 \\ \theta_1 \\ U_2 \\ W_2 \\ \theta_2 \end{array} \right\} = \left[\begin{array}{cccccc} \cos(\gamma) & \sin(\gamma) & 0 & 0 & 0 & 0 \\ -\sin(\gamma) & \cos(\gamma) & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos(\gamma) & \sin(\gamma) & 0 \\ 0 & 0 & 0 & -\sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right] \left\{ \begin{array}{c} \mathcal{U}_1 \\ \mathcal{W}_1 \\ \theta_1 \\ \mathcal{U}_2 \\ \mathcal{W}_2 \\ \theta_2 \end{array} \right\} \tag{17}$$

In symbolic form this equation becomes have,

$$\{D\} = [T]^T \{\mathcal{D}\} \tag{18}$$

Looking at equation 6 and this equation we see how we may transform our beam to a physical coordinate system this equation may be written as,

$$[K]\{D\} = \{F\} \tag{19}$$

Where,

$$[K] \Longrightarrow \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \tag{20}$$

$$\{D\} \Longrightarrow \begin{Bmatrix} U_1^e \\ W_1^e \\ \theta_1^e \\ U_2^e \\ W_2^e \\ \theta_2^e \end{Bmatrix} \tag{21}$$

$$\{F\} \Longrightarrow \begin{Bmatrix} \frac{a_0 L}{2} + \frac{a_1 L^2}{6} \\ \frac{q_1 3L^2}{20} + \frac{q_0 L}{2} \\ \frac{q_1 L^3}{30} + \frac{q_0 L^2}{12} \\ \frac{a_0 L}{2} + \frac{a_1 L^2}{3} \\ \frac{q_1 7L^2}{20} + \frac{q_0 L}{2} \\ -\frac{q_1 L^3}{20} - \frac{q_0 L^2}{12} \end{Bmatrix} + \begin{Bmatrix} \mathcal{P}_1^e \\ \mathcal{V}_l^e \\ \mathcal{M}_l^e \\ \mathcal{P}_2^e \\ \mathcal{V}_r^e \\ \mathcal{M}_r^e \end{Bmatrix} \tag{22}$$

Since we need our matrix equation cast into physical coordinates we use our transformation matrix to obtain,

$$[K][T]^T\{\mathcal{D}\} = \{F\} \tag{23}$$

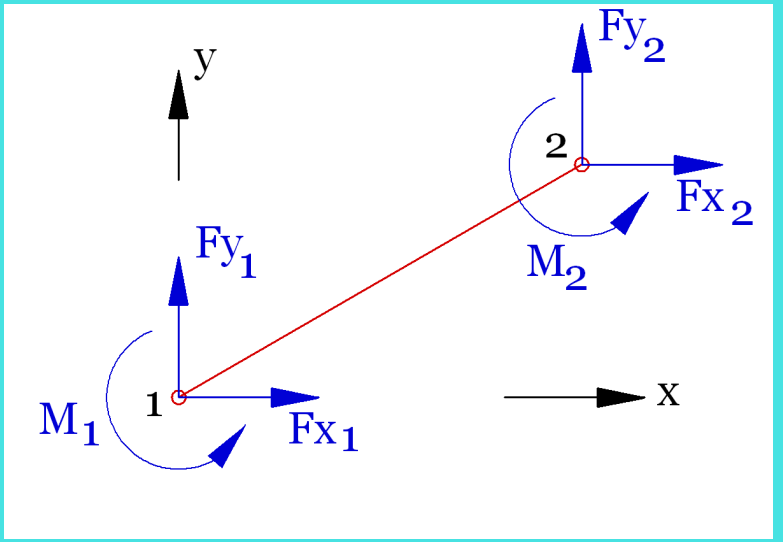
We also notice that our right hand side of this equation is in beam coordinates as well, $\mathcal{P}_1^e$, $\mathcal{V}_l^e$, $\mathcal{M}_l^e$, etc. Therefore the right hand side may be transformed exactly like the displacements.

$$[K][T]^T\{\mathcal{D}\} = [T]^T\{\mathcal{F}\} \tag{24}$$

Where

- $\{F\}$ is in beam coordinates.
- $\{\mathcal{F}\}$ is in physical coordinates.

This means our forces are in the physical coordinates...



I.E.

$$\begin{Bmatrix} Fx_1^e \\ Fy_1^e \\ M_1^e \\ Fx_2^e \\ Fy_2^e \\ M_2^e \end{Bmatrix} = \begin{bmatrix} \cos(\gamma) & -\sin(\gamma) & 0 & 0 & 0 & 0 \\ \sin(\gamma) & \cos(\gamma) & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos(\gamma) & -\sin(\gamma) & 0 \\ 0 & 0 & 0 & \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \mathcal{P}_1^e \\ \mathcal{V}_l^e \\ \mathcal{M}_l^e \\ \mathcal{P}_2^e \\ \mathcal{V}_r^e \\ \mathcal{M}_r^e \end{Bmatrix} \tag{25}$$

As we have noticed before, the inverse of the transformation matrix is equal to it transpose. Therefore our equation may be written as,

$$[T]\,[K]\,[T]^T\,\{\mathcal{D}\} = \{\mathcal{F}\} \tag{26}$$

Or

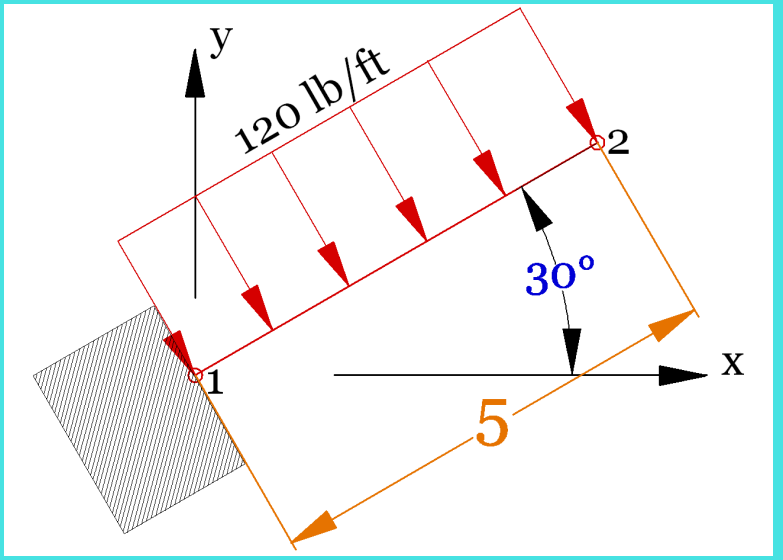
$$[\mathcal{K}]\,\{\mathcal{D}\} = \{\mathcal{F}\} \tag{27}$$

Where

$$[\mathcal{K}] = [T]\,[K]\,[T]^T \tag{28}$$

1.4. Case 01

If we look a simple case,



The transformation matrix is,

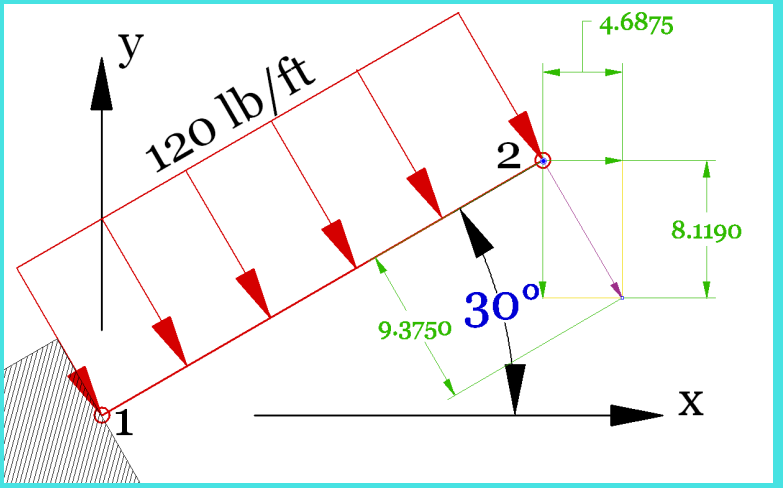
$$[T] \Longrightarrow \begin{bmatrix} 0.8660 & -0.5000 & 0 & 0 & 0 & 0 \\ 0.5000 & 0.8660 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.8660 & -0.5000 & 0 \\ 0 & 0 & 0 & 0.5000 & 0.8660 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{29}$$

Applying this transformation to our stiffness matrix we get,

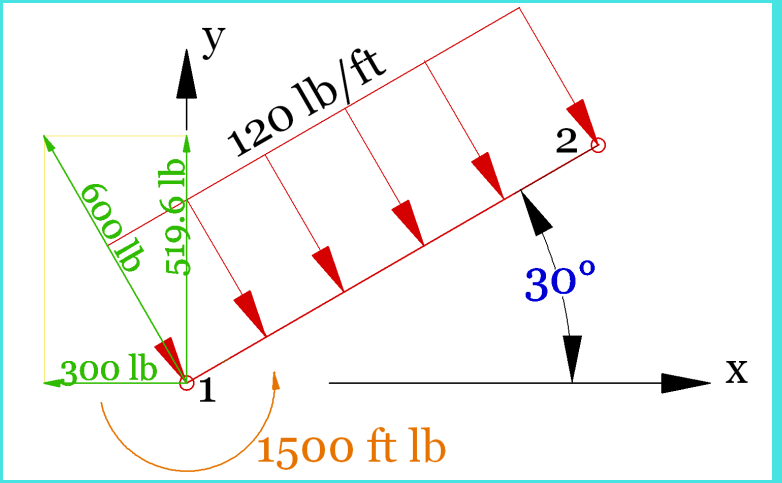
$$[\mathcal{K}] \Longrightarrow \begin{bmatrix} 39.0000 & -32.9090 & -120.0000 & -39.0000 & 32.9090 & -120.0000 \\ -32.9090 & 77.0000 & 207.8461 & 32.9090 & -77.0000 & 207.8461 \\ -120.0000 & 207.8461 & 800.0000 & 120.0000 & -207.8461 & 400.0000 \\ -39.0000 & 32.9090 & 120.0000 & 39.0000 & -32.9090 & 120.0000 \\ 32.9090 & -77.0000 & -207.8461 & -32.9090 & 77.0000 & -207.8461 \\ -120.0000 & 207.8461 & 400.0000 & 120.0000 & -207.8461 & 800.0000 \end{bmatrix} \tag{30}$$

We notice now that there is a dependence between the deflections in the x and y coordinates. See Octave Files.

The deflections for this case are,



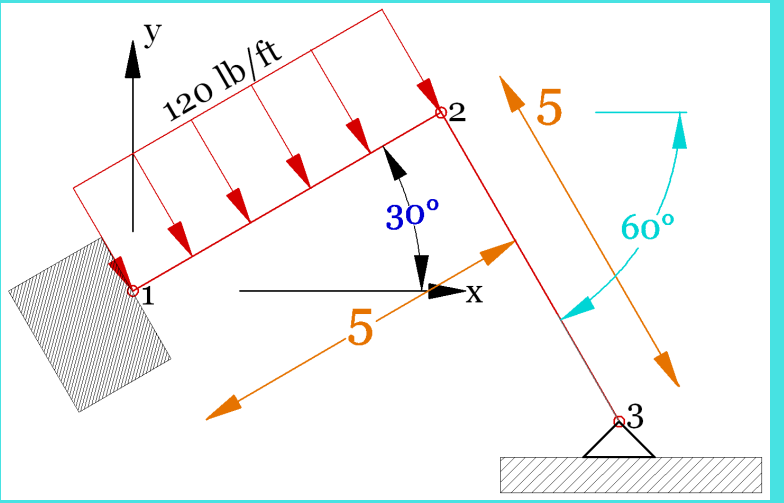
The reactions on the left end of our rotated beam is given as,



Which are correct.

1.5. Case 2

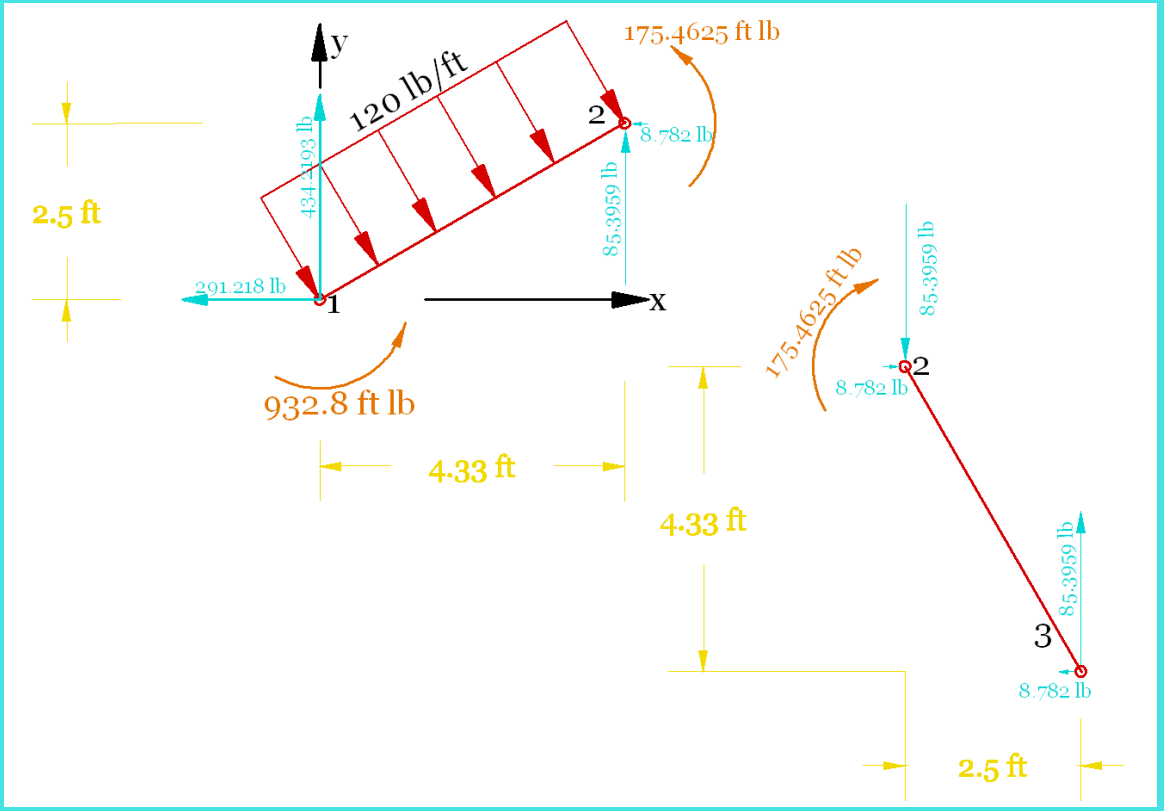
For case 2 we consider a 2 element system.



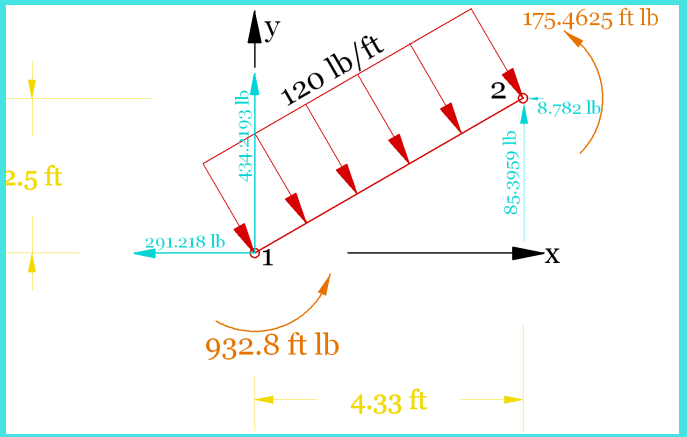
After going through an analysis the reactions on each elements via,

$$[\mathcal{K}^e]\{\mathcal{D}^e\} = \{\mathcal{F}^e\} \tag{31}$$

are,


y
x
120 lb/ft
175.4625 ft lb
2
8.782 lb
434.393 lb
85.3959 lb
2.5 ft
291.218 lb
1
932.8 ft lb
4.33 ft
4.33 ft
175.4625 ft lb
85.3959 lb
2
8.782 lb
3
85.3959 lb
8.782 lb
2.5 ft

For element 1 we have,



If we summed forces in the x directions each would give,

$$\sum F_x = -291.218 - 8.782 + 120 \times 5 \times \sin(30^o) \rightarrow 0 \tag{32}$$

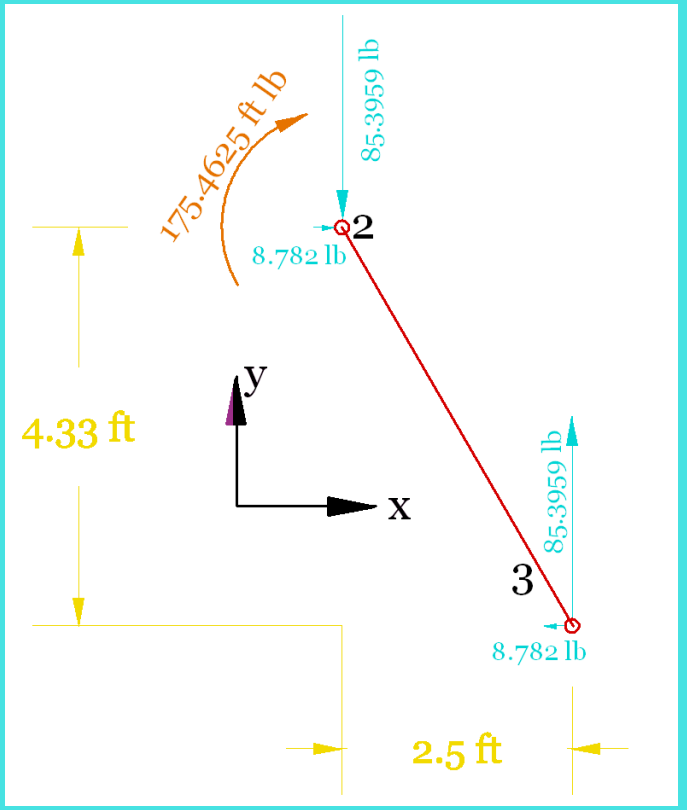
If we summed forces in the y directions each would give,

$$\sum F_y = 434.2193 + 85.3959 - 120 \times 5 \times \cos(30^o) \rightarrow 0 \tag{33}$$

If we summed moments about node 1 we would obtain,

$$\sum M_1 = 932.8 - 120 \times 5 \times 2.5 + 85.3959 \times 4.33 + 8.782 \times 2.5 + 175.4625 \rightarrow 0 \tag{34}$$

For element 2 we have,



If we summed forces in the x and y directions each would give zero as a result. If we summed moments about node 3 we would obtain,

$$\sum M_3 = -175.4625 - 8.782 \times 4.33 + 85.3959 \times 2.5 \rightarrow 0 \tag{35}$$

This information is correct HOWEVER! This information is not what an engineer would be truly interested in. What an engineer would be interest in is, the shear and moment diagrams in each member! That way we could calculate the maximum stress in the beam-bar!

To this end, these global forces can be transformed into moment and shears for our beam and axial loads for our bar. This is done by the following,

$$\{F\} = [T]^T \{\mathcal{F}\} \tag{36}$$

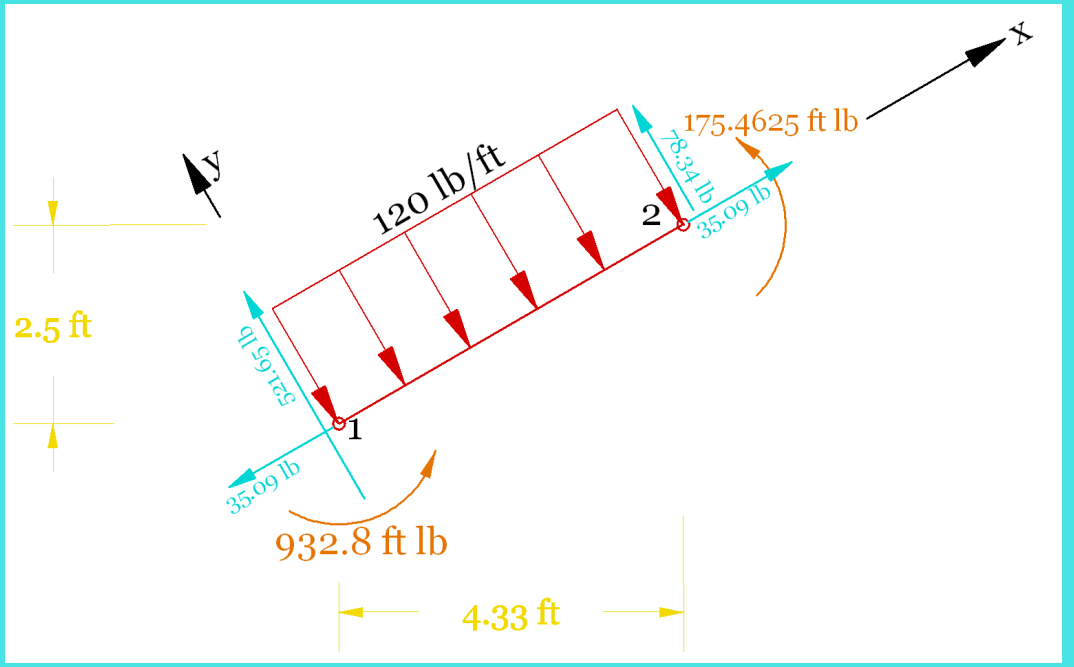
$$\begin{Bmatrix} \mathcal{P}_1^e \\ \mathcal{V}_l^e \\ \mathcal{M}_l^e \\ \mathcal{P}_2^e \\ \mathcal{V}_r^e \\ \mathcal{M}_r^e \end{Bmatrix} = \begin{bmatrix} \cos(\gamma) & \sin(\gamma) & 0 & 0 & 0 & 0 \\ -\sin(\gamma) & \cos(\gamma) & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos(\gamma) & \sin(\gamma) & 0 \\ 0 & 0 & 0 & -\sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} Fx_1^e \\ Fy_1^e \\ M_1^e \\ Fx_2^e \\ Fy_2^e \\ M_2^e \end{Bmatrix} \tag{37}$$

If, for example, we took element 1 on our last case and wished to find the shear, axial force and moment on that member we would have,

$$\begin{Bmatrix} \mathcal{P}_1^e \\ \mathcal{V}_l^e \\ \mathcal{M}_l^e \\ \mathcal{P}_2^e \\ \mathcal{V}_r^e \\ \mathcal{M}_r^e \end{Bmatrix} = \begin{bmatrix} \cos(30) & \sin(30) & 0 & 0 & 0 & 0 \\ -\sin(30) & \cos(30) & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos(30) & \sin(30) & 0 \\ 0 & 0 & 0 & -\sin(30) & \cos(30) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} -291.2180 \\ 434.2193 \\ 932.8074 \\ -8.7820 \\ 85.3959 \\ 175.4625 \end{Bmatrix} \tag{38}$$

OR,

$$\begin{Bmatrix} \mathcal{P}_1^e \\ \mathcal{V}_l^e \\ \mathcal{M}_l^e \\ \mathcal{P}_2^e \\ \mathcal{V}_r^e \\ \mathcal{M}_r^e \end{Bmatrix} = \begin{Bmatrix} -35.0925 \\ 521.6540 \\ 932.8074 \\ 35.0925 \\ 78.3460 \\ 175.4625 \end{Bmatrix} \tag{39}$$



Like wise for element 2 the shear-moment axial loads are,

$$\left\{\begin{array}{c}\mathcal{P}_1^e\\ \mathcal{V}_l^e\\ \mathcal{M}_l^e\\ \mathcal{P}_2^e\\ \mathcal{V}_r^e\\ \mathcal{M}_r^e\end{array}\right\} = \left\{\begin{array}{c}78.3460\\ -35.0925\\ -175.4625\\ -78.3460\\ 35.0925\\ -0.0000\end{array}\right\} \tag{40}$$

