Engineering Vibrations

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Chapter 1

Free Vibration

In this chapter we begin with the concept of free vibration. Mechanical Vibrations influence a wide area of everyday life. This influence may be subtle or drastic depending on the situation. Structural vibrations occur when any object if flexible. This flexibility concept is crucial in considering a vibrating system. Since all object are flexible and posses mass, all object will vibrate under an initial displacement or velocity. Common examples include automobiles, planes, bridges, ships, space vehicles, walk ways in malls, buildings, roads, and the list goes on and on. All mechanical items designed will undergo vibration at one or more points in their service life. In many instances failure to consider vibrational loads may results in catastrophic effects, even loss of life. If you wish to perform an interesting experiment, go into a bookstore located on an upper floor of a local mall. If possible locate a bookstore on this upper floor, the books will add considerable mass to the system. Once there, stand still and pay close attention to the low frequency vibration as people walk by. Adding all the mass of the books to the store lowers the frequency of vibration because the stiffness of the floor remained the same. As we shall see later, this frequency is proportional to the square root of the stiffness over the mass, increasing mass will lower the natural frequency.

1.1 Vibrational Elements

Typically forces on a particle or systems are proportional to three components, displacement, velocity, and acceleration. This dependency can be very complex. However, for the moment we shall only consider linear forms of this dependency.

Forces which are dependent on the displacement usually come from the use of a linear spring. By nature springs are nonlinear if compressed or stretched enough. I am sure you have taken the spring out of your ball point pen and pulled it until it

gave way. During the initial stages of the deflection the force deflection diagram is linear. However, as the spring is continually stretched the force deflection diagram becomes nonlinear. This is shown in the figure, as the deflection increases the spring softens, creating a nonlinear force deflection diagram.

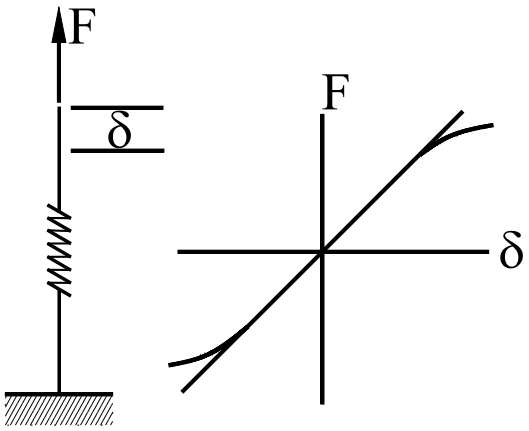


Figure 1.1: Spring Relationship

Forces that depend on the velocity are common. In fact all automobile have these as shock absorbers. Of course there are two elements on an automobiles suspension, a spring and a shock. The spring is simple a force-displacement item, as discussed above. The shock absorber is a force-velocity element. This element will dissipate energy thereby removing the motion. One test of a shock's ability can be seen by pushing down on an automobiles front end several time until the front end is oscillating. Then after the motion is of large enough amplitude stop pushing down on the automobile's front end. After you have taken your hands off

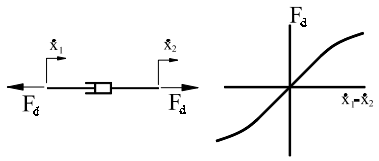


Figure 1.2: Damper Relationship

the car, notice how many oscillations it takes before the automobile is no longer moving. The longer it takes for the motion to stop, the older the shocks. In fact, most garages use this technique to determine if the shocks need to be replaced. The purpose of the shocks are to dissipate the energy of vibration. Otherwise, any bump the automobile would hit will cause a vibration that would continue for several minutes. Shock absorbers are simple elements which have a force-velocity relationship. This relationship is created by a fluid inside a piston-cylinder arrangement. Under normal operating conditions the force-velocity relationship is linear. Since the primary duty of the shock absorber is the removal of energy from the system, it would be expected that this energy would manifest itself in some other

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form via the first law of thermodynamics. The first law is fulfilled by an increase in temperature of the fluid in the shock absorber which in turn will drop the viscosity of the working fluid. Since the viscosity is the primary mechanism for energy absorption one would expect that requiring the shock absorber to dissipate energy too quickly would give rise to a non linear force-velocity relationship. These are all consideration which an engineer should be take into account during a design of a dynamic system.

1.2 Spring Mass System

In this section we investigate a common vibrational system, the spring mass system. This common system forms the bases for later more complicated systems. The system, shown in figure 1.3, is made up of a mass on frictionless wheels at-

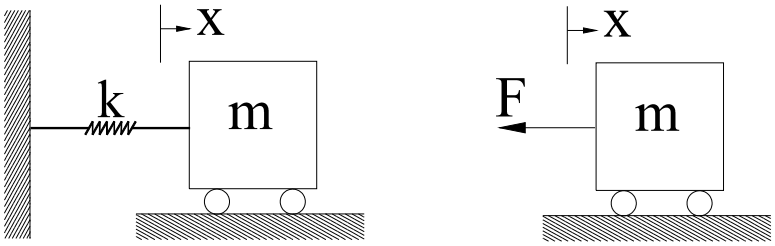


Figure 1.3: Spring Mass System

tached to a fixed wall via a linear spring. On the right of this figure a free body diagram is drawn. Using this diagram as a reference, we use Newton's second law,

$$\sum \overrightarrow{F} = m\overrightarrow{a} \tag{1.1}$$

making note the force from the spring is kx and the deflection to the right is defined as positive. Hence the equation becomes,

$$\sum \overrightarrow{F} = -kx\left(t\right) = m\ddot{x}\left(t\right) \tag{1.2}$$

Rearranging this equation we get,

$$m\ddot{x}(t) + kx(t) = 0 \tag{1.3}$$

There are several approaches one can use to develop a solution to this differential equation. recall from differential equation the assumed form of the solution,

$$x(t) = Ae^{\lambda t} \tag{1.4}$$

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Using this form of a solution, its derivatives $\dot{x}(t) = A\lambda e^{\lambda t}$ and $\ddot{x}(t) = A\lambda^2 e^{\lambda t}$ we can determine the value of coefficient λ by placing these into our differential equation.

$$mA\lambda^2 e^{\lambda t} + kAe^{\lambda t} = 0 \tag{1.5}$$

Since this $Ae^{\lambda t}$ is not zero it can be divided out to give the characteristic equation,

$$m\lambda^2 + k = 0 \tag{1.6}$$

Solving for the constant λ we get,

$$\lambda = \pm\sqrt{-\frac{k}{m}} = \pm i\sqrt{\frac{k}{m}} = \pm i\omega \tag{1.7}$$

Where the $i = \sqrt{-1}$ is an imaginary number. Since there are two roots, λ's, to this equation there exist two different functions which satisfy the differential equation. Therefore, the solution is a linear combination of these two functions. The solution can be expressed as,

$$x(t) = A_1 e^{i\omega t} + A_2 e^{-i\omega t} \tag{1.8}$$

Where A_1 and A_2 are complex constants whose value is determined by the initial values. We now remember Euler's identity,

$$\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \\ e^{-i\theta} &= \cos\theta - i\sin\theta \end{aligned} \tag{1.9}$$

Using these relationships in the solution we get ,

$$x(t) = A_1\left(\cos\omega t + i\sin\omega t\right) + A_2\left(\cos\omega t - i\sin\omega t\right) \tag{1.10}$$

Expressing the complex constants as,

$$\begin{aligned} A_1 &= a_{11} + ia_{12} \\ A_2 &= a_{21} + ia_{22} \end{aligned} \tag{1.11}$$

Expressing this out in a complete form back into the solution we obtain,

$$x(t) = \left[a_{11} + a_{21} + i\left(a_{12} + a_{22}\right)\right]\cos\omega t + \left[a_{22} - a_{12} + i\left(a_{11} - a_{21}\right)\right]\sin\omega t \tag{1.12}$$

Logic would tell us the constants $a_{12} + a_{22}$ and $a_{11} - a_{21}$ must be zero. Otherwise, how can we get an imaginary solution to a real problem. From these two equations we notice our complex constants are actually complex conjugates of one another, $A_1 = a_{11} + ia_{12}$ and $A_2 = a_{11} - ia_{12}$. Rearranging, the complete solution becomes,

$$x(t) = A_s\sin\omega t + A_c\cos\omega t \tag{1.13}$$

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1.2.1 Phasors

Taking **any** function of the form in equation 1.13, it is convenient to write this function in either of the two forms,

$$\begin{aligned} x(t) &= A_0 \cos(\omega t - \phi) \\ x(t) &= A_0 \sin(\omega t + \varphi) \end{aligned} \tag{1.14}$$

Doing so may seem to simplify the expression, but developing equations 1.14 from equation 1.13 may not seem immediately apparent. Consider for a moment figure 1.4 where two orthogonal vectors are rotating counterclockwise at a constant rate, ω. As these vectors rotate counterclockwise their y component sweeps out a sine

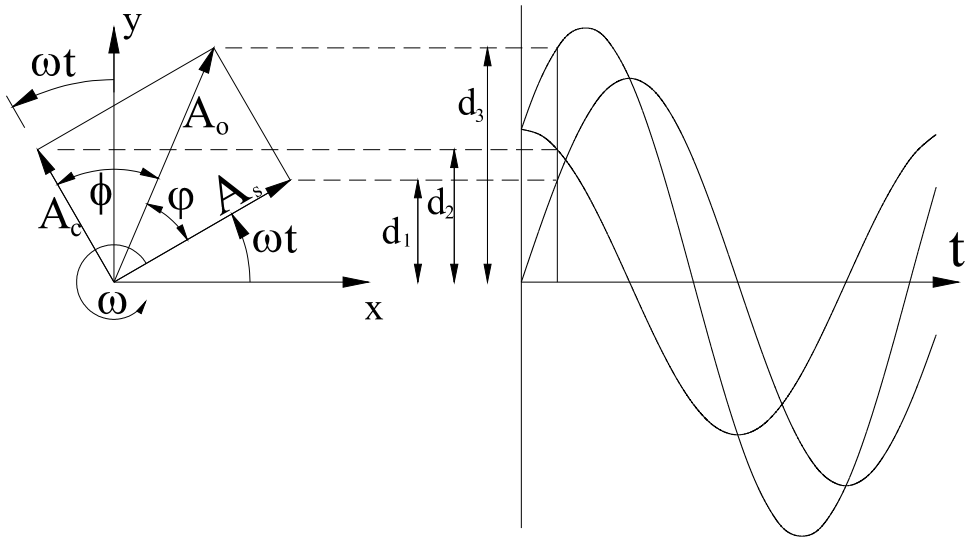


Figure 1.4: Phasor Vector Plots

or cosine wave along the time axis. Careful examination of this figure revels the distance $d_1 = A_s \sin \omega t$ and $d_2 = A_c \cos \omega t$. Examination also revels $d_3 = d_1 + d_2$ for any time t. Therefore, d_3 is the same as $x\,(t)$ from equation 1.13. Notice this addition, $d_3 = d_1 + d_2$, on the right of the figure is equivalent to the vector addition on the left diagram, phasor diagram. This vector addition on revels the magnitude of the resultant vector is

$$\boxed{A_0 = \sqrt{A_s^2 + A_c^2}} \tag{1.15}$$

We now have two choices as to how to represent this as a single sine of cosine function. If we relate this A_0 vector to the x axis we will obtain the sine representation

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of this function. Doing so we see the angle between the x axis and the vector A_0 is $(\omega t + \varphi)$ giving the functional representation as,

$$\boxed{x(t) = A_0 \sin(\omega t + \varphi)} \tag{1.16}$$

Where the phase angle is,

$$\boxed{\varphi = \tan^{-1}\left(\frac{A_c}{A_s}\right)} \tag{1.17}$$

The angle between the y axis and the vector A_0 is $(\omega t - \phi)$. Relating the vector's angle to the y axis we obtain the cosine representation to this function as,

$$\boxed{x(t) = A_0 \cos(\omega t - \phi)} \tag{1.18}$$

In this case the phase angle becomes,

$$\boxed{\phi = \tan^{-1}\left(\frac{A_s}{A_c}\right)} \tag{1.19}$$

Depending on the particular situation it may be desirable to use the sine over the cosine representation of the function, or vice versa.

1.2.2 Initial Conditions

We return to equation 1.13 to evaluate the coefficients A_s and A_c based on the initial conditions. Remember from differential equations that a second order differential equation requires two initial conditions. These two conditions are the initial position, x_0, and the initial velocity, v_0. The initial position, of course, occurs at time equal to zero giving,

$$x(0) = x_0 = A_s \sin 0 + A_c \cos 0 = A_c \tag{1.20}$$

Taking the derivative of equation 1.13 at time equal to zero we get,

$$\dot{x}(0) = v_0 = A_s \omega \cos 0 - A_c \omega \sin 0 = A_s \omega \tag{1.21}$$

hence the complete solution becomes,

$$\boxed{x(t) = \frac{v_0}{\omega} \sin \omega t + x_0 \cos \omega t} \tag{1.22}$$

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According to the previous section this equation can be written in either of the following forms,

$$\boxed{x(t) = \sqrt{\frac{v_0^2}{\omega^2} + x_0^2}\ \sin\left(\omega t + \tan^{-1}\left(\frac{x_0\omega}{v_0}\right)\right)} \tag{1.23}$$

$$\boxed{x(t) = \sqrt{\frac{v_0^2}{\omega^2} + x_0^2}\ \cos\left(\omega t - \tan^{-1}\left(\frac{v_0}{x_0\omega}\right)\right)} \tag{1.24}$$

All of these boxed equations are the same functions as proven in the last section.

Example 1.2.1

Here we take an example system with a spring constant of 400 Nt/m. The mass of the system is 4 kg. From equation 1.7 we get the natural frequency of vibration as,

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{400 Nt/m}{4kg}} = 10 rad/sec \tag{1.25}$$

A Nt is the same as a kg m /s^2, in addition radians are unitless.

The initial position of the mass is $x_0 = .5$ m, the initial velocity is $v_0 = 2$ m/s. Therefore, the amplitude of the vibration becomes,

$$A_0 = \sqrt{\frac{v_0^2}{\omega^2} + x_0^2} = \sqrt{\frac{(2 m/sec)^2}{(10 rad/sec)^2} + (0.5m)^2} = 0.538516m \tag{1.26}$$

The final information needed is the phase angle. At this point we must which we prefer, sine or cosine. For no particular reason we shall use the sine, noting that the function is in fact the same. The phase angle for the sine is,

$$\varphi = \tan^{-1}\left(\frac{x_0\omega}{v_0}\right) = \tan^{-1}\left(\frac{0.5m\ 10 rad/sec}{2 m/sec}\right) = 1.19029\ rad \tag{1.27}$$

Combining all this information together we obtain the complete solution as,

$$x(t) = 0.538516\ \sin\left(10t + 1.19029\right) \tag{1.28}$$

This equation is shown in figure 1.5. The time between two peaks of the sine wave is called the period and is defined by,

$$\boxed{T = \frac{2\pi}{\omega}\ (sec)} \tag{1.29}$$

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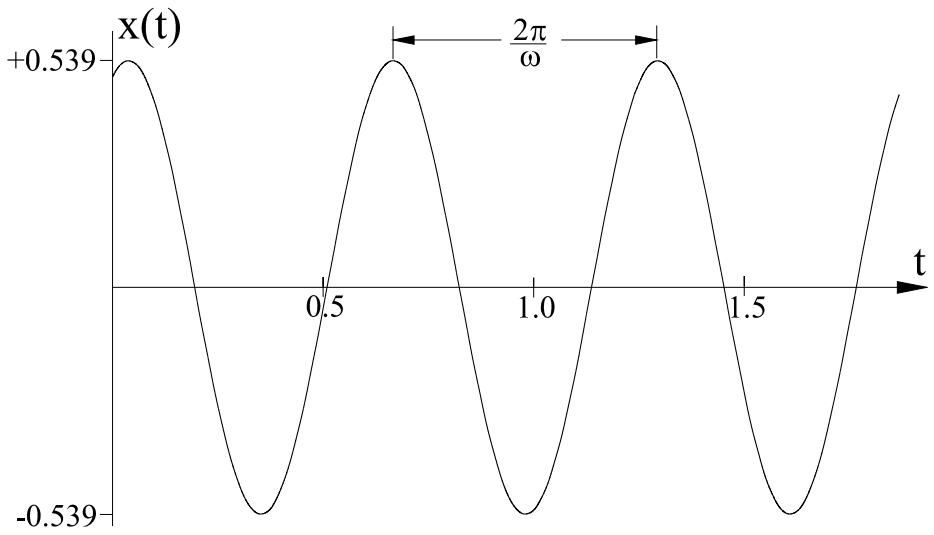


Figure 1.5: Example Solution

This equation comes from the fact that in order to complete a cycle of a sine wave the angle must sweep through 2π radians. Since the sweep angle is ωt, it would make sense that making these two terms equal will give us the time required to complete the one cycle. Or, $\omega T = 2\pi$, where T is used instead of t signifying its special significance. In the present case this period is,

$$T = \frac{2\pi}{10} = .6283 \text{ (sec)} \tag{1.30}$$

Often another term is used to reference the frequency of vibration. This term is called Hertz, or cycles per second. this term is used almost exclusively in the audio reproduction field. Look on any data sheet for an audio component an the frequency response curves of specifications will reference the frequency in Hertz. Since T is the time for one cycle to complete, its units should be in seconds per cycle. Therefore, the inverse of these units will be cycles per second, or Hertz. In equation form this becomes,

$$\boxed{f \text{ (Hz)} = \frac{1}{T} = \frac{\omega}{2\pi} \text{ (Hz)}} \tag{1.31}$$

In our example this frequency becomes,

$$f \text{ (Hz)} = \frac{1}{T} = 1.59 \text{(Hz)} \tag{1.32}$$

This Completes The Example

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1.3 Viscous Damping

We now consider the addition a a damper to the system as shown in figure 1.6. The system configuration is shown on the left of this figure while the free body diagram of this system is shown to the right of this figure. The system's components, both spring and damper, are considered linear. The spring force, F_s, is linearly

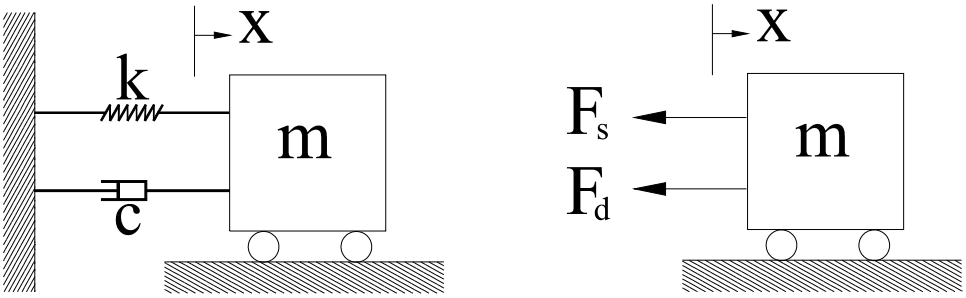


Figure 1.6: Spring Mass Damper System

dependent on the displacement through $F_s = kx$. Likewise, the damper is linearly dependent on the velocity through $F_d = c\dot{x}$. Using Newton's second law we obtain the following equation,

$$\begin{aligned} \sum \overrightarrow{F} &= m\overrightarrow{a} \\ -F_s - F_d &= -kx - c\dot{x} = m\ddot{x} \end{aligned} \tag{1.33}$$

Rearranging the last equation we get a second order differential equation

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{1.34}$$

Assuming a solution of the form,

$$\begin{aligned} x(t) &= Ae^{\lambda t} \\ \dot{x}(t) &= A\lambda e^{\lambda t} \\ \ddot{x}(t) &= A\lambda^2 e^{\lambda t} \end{aligned} \tag{1.35}$$

Inserting these terms into our differential equation we get,

$$mA\lambda^2 e^{\lambda t} + cA\lambda e^{\lambda t} + kAe^{\lambda t} = 0 \tag{1.36}$$

Since the term A and $e^{\lambda t}$ are not always zero we can remove them from our consideration to obtain a characteristic equation.

$$m\lambda^2 + c\lambda + k = 0 \tag{1.37}$$

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The solution to this equation becomes, through use of the quadratic equation,

$$\lambda = -\frac{c}{2m} \pm \frac{\sqrt{c^2 - 4mk}}{2m} \tag{1.38}$$

Rearranging this equation into a more convenient form we obtain,

$$\begin{aligned} \lambda &= -\frac{c}{2m} \pm \frac{2\sqrt{mk}}{2m}\sqrt{\frac{c^2}{4mk} - 1} \\ &= -\frac{c}{2m} \pm \sqrt{\frac{k}{m}}\sqrt{\frac{c^2}{4mk} - 1} \end{aligned} \tag{1.39}$$

Using the undamped natural frequency,

$$\boxed{\omega_n = \sqrt{k/m}.} \tag{1.40}$$

We now define a new variable,

$$\boxed{\zeta = \frac{c}{2m\omega_n}} \tag{1.41}$$

If we square and expand this term we see the term under the radical.

$$\zeta^2 = \frac{c^2}{4m^2\omega_n^2} = \frac{c^2 m}{4m^2 k} = \frac{c^2}{4mk} \tag{1.42}$$

Combining equations 1.39, 1.40, 1.41and 1.42 we obtain the two roots to our equation,

$$\boxed{\lambda = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}} \tag{1.43}$$

This equation gives three distinct results. These results are written in equation form as,

$$\begin{aligned} \lambda &= -\zeta\omega_n \pm i\omega_n\sqrt{1 - \zeta^2} && \text{for } 0 < \zeta < 1 \\ \lambda &= -\zeta\omega_n && \text{for } \zeta = 1 \\ \lambda &= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} && \text{for } \zeta > 1 \end{aligned} \tag{1.44}$$

Where care was taken such that the sign of the term under the radical is always positive. Looking back at the differential equation, 1.34, we divide by the mass to obtain,

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0 \tag{1.45}$$

Using equations 1.40 and 1.41 the differential equation can be rewritten as,

$$\boxed{\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0} \tag{1.46}$$

Which is a general form of this equation we shall use in later sections and chapters. Since this equation gives three distinctly different solution we shall devote a section to each.

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1.4 Under damped Case ($0 < \zeta < 1$)

Referring back to equation 1.44 we take the first case listed. Under the restriction of $0 < \zeta < 1$ we return to the original assumed solution to obtain,

$$x(t) = A_1 e^{-\zeta\omega_n t + i\omega_n\sqrt{1-\zeta^2}t} + A_2 e^{-\zeta\omega_n t - i\omega_n\sqrt{1-\zeta^2}t} \tag{1.47}$$

Which after a quick rearrangement becomes,

$$x(t) = e^{-\zeta\omega_n t}\left(A_1 e^{+i\omega_n\sqrt{1-\zeta^2}t} + A_2 e^{-i\omega_n\sqrt{1-\zeta^2}t}\right) \tag{1.48}$$

Making a correlation between the present equation 1.48 and how equation 1.13 was developed from equation 1.8 we can conclude our solution is,

$$x(t) = e^{-\zeta\omega_n t}\left[A_s \sin\left(\omega_n\sqrt{1-\zeta^2}t\right) + A_c \cos\left(\omega_n\sqrt{1-\zeta^2}t\right)\right] \tag{1.49}$$

As we covered in the section on rotating vectors this equation can be written as,

$$\boxed{\begin{aligned} x(t) &= e^{-\zeta\omega_n t}\left\{A_s \sin(\omega_d t) + A_c \cos(\omega_d t)\right\} \\ &= A_0 e^{-\zeta\omega_n t}\sin(\omega_d t + \varphi) \\ &= A_0 e^{-\zeta\omega_n t}\cos(\omega_d t - \phi) \end{aligned}} \tag{1.50}$$

Where,

$$\boxed{\omega_d = \omega_n\sqrt{1-\zeta^2}} \tag{1.51}$$

is called the damped natural frequency. There are still two undetermined constants. Taking the first representation of the solution we obtain,

$$\begin{aligned} x(t) =& e^{-\zeta\omega_n t}\left\{A_s \sin(\omega_d t) + A_c \cos(\omega_d t)\right\} \\ \dot{x}(t) =& e^{-\zeta\omega_n t}\left\{A_s\omega_d \cos(\omega_d t) - A_c\omega_d \sin(\omega_d t)\right\} \\ & - \zeta\omega_n e^{-\zeta\omega_n t}\left\{A_s \sin(\omega_d t) + A_c \cos(\omega_d t)\right\} \end{aligned} \tag{1.52}$$

Determining the constants based on the initial conditions of the problem give way to the following equations,

$$\begin{aligned} x(0) = x_0 =& A_c \\ \dot{x}(0) = v_0 =& A_s\omega_d - \zeta\omega_n A_c \end{aligned} \tag{1.53}$$

The Solution for the constants become,

$$\boxed{\begin{aligned} A_c &= x_0 \\ A_s &= \frac{v_0 + \zeta\omega_n x_0}{\omega_d} \end{aligned}} \tag{1.54}$$

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From equation 1.50 and the section on rotating vectors we obtain the magnitude of the vibration as,

$$\boxed{A_0 = \frac{\sqrt{(x_0\omega_d)^2 + (v_0 + \zeta\omega_n x_0)^2}}{\omega_d}} \tag{1.55}$$

In addition we obtain the phase angles for sine and cosine respectively as,

$$\boxed{\varphi = \tan^{-1}\left(\frac{x_0\omega_d}{v_0 + \zeta\omega_n x_0}\right)} \tag{1.56}$$

$$\boxed{\phi = \tan^{-1}\left(\frac{v_0 + \zeta\omega_n x_0}{x_0\omega_d}\right)} \tag{1.57}$$

Example 1.4.1

Assume we have a spring mass damper system shown in figure 1.6 with the constants defined as,

$$\begin{aligned} m &= 20 \ \ kg \\ k &= 5 \ \ Nt/m \\ c &= 5 \ \ Nt\ sec/m \\ x_0 &= 1.0 \ \ m \\ v_0 &= 3.0 \ \ m \end{aligned} \tag{1.58}$$

With this data, we first calculate the quantities relating to the system,

$$\begin{aligned} \omega_n &= \sqrt{\frac{k}{m}} = \sqrt{\frac{5}{20}} = \frac{1}{2} = 0.5 \\ \zeta &= \frac{c}{2m\omega_n} = \frac{1}{4} = 0.25 \\ \omega_d &= \omega_n\sqrt{1-\zeta^2} = \frac{\sqrt{15}}{8} = 0.484123 \end{aligned} \tag{1.59}$$

Now we calculate the quantities relating to the imposed initial conditions, first the initial magnitude of the vibration,

$$\begin{aligned} A_0 &= \frac{\sqrt{(x_0\omega_d)^2 + (v_0 + \zeta\omega_n x_0)^2}}{\omega_d} \\ &= \frac{\sqrt{\left(\frac{\sqrt{15}}{8}\right)^2 + \left(3 + \frac{1}{8}\right)^2}}{\frac{\sqrt{15}}{8}} = 8\sqrt{\frac{2}{3}} = 6.53197 \end{aligned} \tag{1.60}$$

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The phase angle related to the sine function is,

$$\varphi = \tan^{-1}\left(\frac{x_0\omega_d}{v_0 + \zeta\omega_n x_0}\right) = \tan^{-1}\left(\frac{\frac{\sqrt{15}}{8}}{3 + \frac{1}{8}}\right) = 0.153698 \tag{1.61}$$

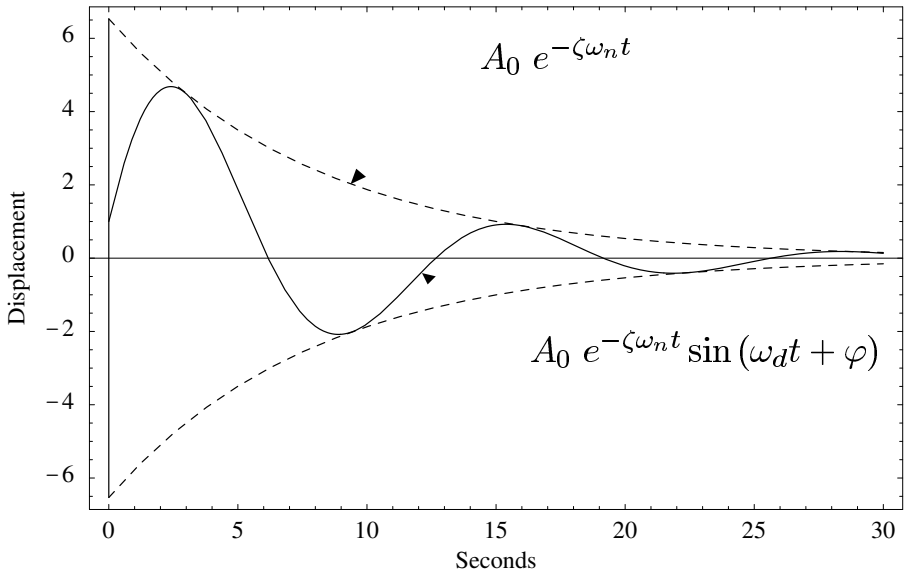


Figure 1.7: Under Damped System Response

Combining all this information together we obtain the displacement as a function of time.

$$x(t) = 6.53197e^{-t/8}\sin(0.484123t + 0.1536975) \tag{1.62}$$

This equation is plotted in figure 1.7. In addition to this function the function $6.53197e^{-t/8}$ *is plotted in dashed lines. This function is in effect an envelope to the response because the sine function can only obtain values between plus and minus one. This exponential decay of the motion can often be used to experimentally determine the damping in a system, as will be describe later.*

This Completes The Example

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1.5 Critically Damped System ($\zeta = 1$)

When $\zeta = 1$, from equation 1.44, we have a repeated root to the characteristic equation. For a repeated root the solution takes the form,

$$x(t) = A_1 e^{-\omega_n t} + A_2 t e^{-\omega_n t} \tag{1.63}$$

In order to determine the constants, as before, we resort to the initial conditions of the problem. Taking equation 1.63 and its differential we obtain the following equations,

$$\begin{aligned} x(t) &= A_1 e^{-\omega_n t} + A_2 t e^{-\omega_n t} \\ \dot{x}(t) &= -A_1 \omega_n e^{-\omega_n t} + A_2 e^{-\omega_n t} - A_2 \omega_n t e^{-\omega_n t} \end{aligned} \tag{1.64}$$

Evaluating these two equations at time equal to zero gives,

$$\begin{aligned} x(0) &= x_0 = A_1 \\ \dot{x}(0) &= v_0 = -A_1 \omega_n + A_2 \end{aligned} \tag{1.65}$$

Solving for A_1 and A_2, placing the result back into equation 1.63 we get a solution to the critically damped system as,

$$\boxed{x(t) = \left(x_0 + \left(v_0 + x_0 \omega_n\right) t\right) e^{-\omega_n t}} \tag{1.66}$$

Example 1.5.1
Consider a system like that shown in figure 1.6 with the constants defined as,

$$\begin{aligned} m &= 20 \ \ kg \\ k &= 5 \ \ Nt/m \\ \zeta &= 1 \\ x_0 &= 1.0 \ \ m \\ v_0 &= -3, 0, +3 \ \ m \end{aligned} \tag{1.67}$$

Here we wish to consider three cases to the see the affect the initial velocity plays on the solution. With this data we calculate the quantities relating to the system,

$$\begin{aligned} \omega_n &= \sqrt{\frac{k}{m}} = \sqrt{\frac{5}{20}} = \frac{1}{2} = 0.5 \\ \zeta &= 1 \end{aligned} \tag{1.68}$$

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Returning to the critically damped solution we obtain

$$x(t) = (x_0 + (v_0 + x_0\omega_n)\, t)\, e^{-\omega_n t} = (1 + (v_0 + 0.5)\, t)\, e^{-.5t} \tag{1.69}$$

Using the three initial velocities to express the three solution, we get, Returning to the critically damped solution we obtain

$$\begin{array}{ll} x(t) = (1 - 2.5t)\, e^{-.5t} & v_0 = -3 \\ x(t) = (1 + 0.5t)\, e^{-.5t} & v_0 = 0 \\ x(t) = (1 + 3.5t)\, e^{-.5t} & v_0 = +3 \end{array} \tag{1.70}$$

A plot of each of these functions is shown in figure 1.8. Notice the response does not oscillate or overshoot the time axis. This should be expected since an exponential function decays without overshoot.

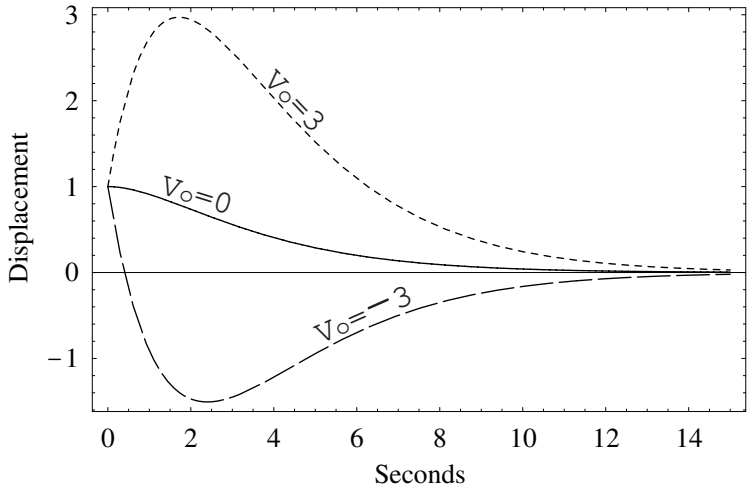


Figure 1.8: Critically Damped System Response

This Completes The Example

1.6 Over Damped System ($\zeta > 1$)

When $\zeta > 1$, from equation 1.44, we have a two real distinct roots to the characteristic equation. For a this situation the solution takes the form,

$$x(t) = e^{-\zeta\omega_n t}\left(A_1 e^{\omega_n\sqrt{\zeta^2-1}t} + A_2 e^{-\omega_n\sqrt{\zeta^2-1}t}\right) \tag{1.71}$$

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The derivative of this equation is,

$$\begin{aligned}\dot{x}(t) =& A_1 \left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right) e^{-\zeta\omega_n t+\omega_n\sqrt{\zeta^2-1}t} + \\ & A_2 \left(-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right) e^{-\zeta\omega_n t-\omega_n\sqrt{\zeta^2-1}t}\end{aligned} \tag{1.72}$$

Equating these two equations to the initial conditions we get,

$$\begin{aligned}x(0) &= x_0 = A_1 + A_2 \\ \dot{x}(0) &= v_0 = A_1 \left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right) + A_2 \left(-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right)\end{aligned} \tag{1.73}$$

This system could be placed in matrix form to aid in its solution. As we shall see in the next chapter, this form will be helpful in obtaining solution to complicated equations.

$$\begin{bmatrix} 1 & 1 \\ -\zeta\omega_n + \omega_n\sqrt{\zeta^2-1} & -\zeta\omega_n - \omega_n\sqrt{\zeta^2-1} \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} x_0 \\ v_0 \end{Bmatrix} \tag{1.74}$$

The determinate of the matrix is, $-2\omega_n\sqrt{\zeta^2-1}$. Now using Cramers rule to find the constants we get,

$$A_1 = \frac{\begin{vmatrix} x_0 & 1 \\ v_0 & -\zeta\omega_n - \omega_n\sqrt{\zeta^2-1} \end{vmatrix}}{-2\omega_n\sqrt{\zeta^2-1}} \tag{1.75}$$

$$A_2 = \frac{\begin{vmatrix} 1 & x_0 \\ -\zeta\omega_n + \omega_n\sqrt{\zeta^2-1} & v_0 \end{vmatrix}}{-2\omega_n\sqrt{\zeta^2-1}} \tag{1.76}$$

Multiplying out these equations we get the solution for the constants in terms of the initial conditions.

$$\boxed{\begin{aligned} A_1 &= \frac{x_0\omega_n\left(\sqrt{\zeta^2-1}+\zeta\right)+v_0}{2\omega_n\sqrt{\zeta^2-1}} \\ A_2 &= \frac{x_0\omega_n\left(\sqrt{\zeta^2-1}-\zeta\right)-v_0}{2\omega_n\sqrt{\zeta^2-1}} \end{aligned}} \tag{1.77}$$

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Therefore, the complete solution becomes,

$$x(t) = \frac{x_0\omega_n\left(\sqrt{\zeta^2-1}+\zeta\right)+v_0}{2\omega_n\sqrt{\zeta^2-1}}\, e^{-\zeta\omega_n t+\omega_n\sqrt{\zeta^2-1}t} + \frac{x_0\omega_n\left(\sqrt{\zeta^2-1}-\zeta\right)-v_0}{2\omega_n\sqrt{\zeta^2-1}}\, e^{-\zeta\omega_n t-\omega_n\sqrt{\zeta^2-1}t} \tag{1.78}$$

Example 1.6.1
Consider, again, a system like that shown in figure 1.6 with the constants defined as,

$$\begin{aligned} m &= 20 \ \ kg \\ k &= 5 \ \ Nt/m \\ c &= 40 \\ &Nt\ sec/m \end{aligned} \tag{1.79}$$

Here we wish to consider three cases to the see the affect the initial velocity plays on the solution. With this data we calculate the quantities relating to the system,

$$\begin{aligned} \omega_n &= \sqrt{\frac{k}{m}} = \sqrt{\frac{5}{20}} = \frac{1}{2} = 0.5 \\ \zeta &= \frac{c}{2m\omega_n} = \frac{40}{20} = 2 \\ \omega_d &= \omega_n\sqrt{\zeta^2-1} = \frac{\sqrt{3}}{2} = 0.866025 \end{aligned} \tag{1.80}$$

With this information we turn now to determining the coefficients in the differential equation.

$$\begin{aligned} A_1 &= 0.57735\left(v_0 + 1.86603x_0\right) \\ A_2 &= -0.57735\left(v_0 + 0.133975x_0\right) \end{aligned} \tag{1.81}$$

From these equations we now investigate the behavior of the solution for several initial conditions given below.

$$\begin{aligned} x_0 &= 1.0 \ \ m \\ v_0 &= -3, 0, +3 \ \ m \end{aligned} \tag{1.82}$$

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Using the three initial velocities and initial displacements we get three solutions, one for each case.

$$\begin{array}{ll} x(t) = 1.6547e^{-1.86603t} - 0.654701e^{-0.133975t} & v_0 = -3 \\ x(t) = -0.0773503e^{-1.86603t} + 1.07735e^{-0.133975t} & v_0 = 0 \\ x(t) = -1.8094e^{-1.86603t} + 2.8094e^{-0.133975t} & v_0 = +3 \end{array} \tag{1.83}$$

A plot of each of these functions is shown in figure 1.9. Like the critically damped system the response does not oscillate or overshoot the time axis. This should be expected since an exponential function decays without overshoot.

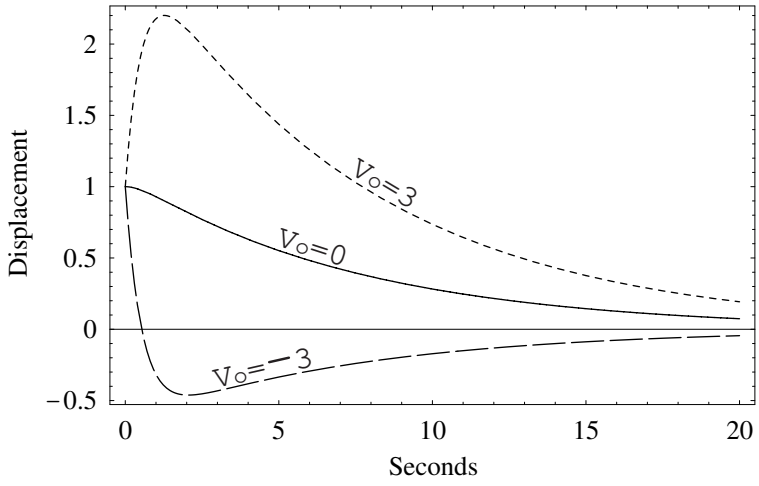


Figure 1.9: Over Damped System Response

This Completes The Example

1.7 System Analogies

In this section dynamic analogies are introduced. In later chapters these analogies will be made in order to develop and solve problems in related fields. In this section, electrical analogies will be explained due to the authors assumption that the reader is already familiar with and has taken a course in electrical circuits. Given a circuit as shown in figure 1.9, where the capacitor is charge to an initial voltage of V_i. Remembering the voltage across each element is given by,

Voltage drop across a resistor	$= IR$
Voltage drop across a capacitor	$= \frac{Q}{C}$
Voltage drop across an inductor	$= L\frac{dI}{dt}$

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The instant the switch is closed the sum of the voltage drops across each element must equal to zero. This sum yields a differential equation,

$$L\frac{dI}{dt} + RI + \frac{Q}{C} = 0 \tag{1.84}$$

Making not of the relationship between the charge Q and the current I, $I = \dot{Q}$, we rewrite our differential equation as,

$$L\ddot{Q} + R\dot{Q} + \frac{Q}{C} = 0 \tag{1.85}$$

Or we could take the time derivative of this equation to get,

$$L\ddot{I} + R\dot{I} + \frac{I}{C} = 0 \tag{1.86}$$

Rewriting this equation in the standard generalized form of our second order equation 1.46, we see our coefficients become,

$$\begin{aligned} \omega_n &= \frac{1}{\sqrt{LC}} \\ \zeta &= \frac{R}{2}\sqrt{\frac{C}{L}} \end{aligned} \tag{1.87}$$

There also exist acoustical, fluids and rotational analogies which will be covered later in this book.

1.8 System Components (Springs)

In this section we shall investigate the variety of elements which act as springs in a vibrational system. We start with springs in recti-linear motion.

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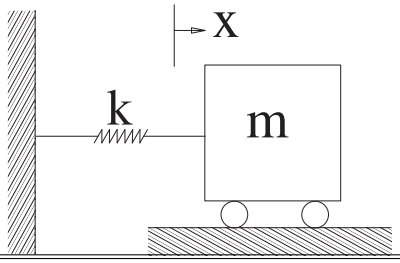
1.9 Homework

Given a system, 1.9 where the spring stiffness is $k = 5000$ lb/ft and the mass is $m = 10$ slugs.
The solution may be represented as:

$$x(t) = A_c \cos(\omega t + \phi) = A_s \sin(\omega t + \theta) = B \sin(\omega t) + C \cos(\omega t)$$

If the initial conditions for this system are $x(t=0) = 2$ inches and $v(t=0) = 5$ inches/sec,

1. What is this systems natural frequency?
2. What is A_c and ϕ?
3. What is A_s and θ?
4. What is A and B?



If $x(t) = A_1 e^{\jmath\omega t} + A_2 e^{-\jmath\omega t}$ and $x(t=0) = 4$ and $v(t=0) = 7$, what is A_1 and A_2?

Write $(10 + 77\jmath)\, e^{\jmath\omega t} + (10 - 77\jmath)\, e^{-\jmath\omega t}$ in terms of $\sin(\omega t)$ and $\cos(\omega t)$

Can the trig identity,

$$\sin(\alpha + \beta) = \sin\alpha \cos\beta + \sin\beta \cos\alpha$$

be proven using euler's identities? If so, show how!

If a 300 pound machine is placed on a flexible support (all supports are flexible to some degree) and the support drops one inch, what is the natural frequency of this system?

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Fill in the table Below. Also use the Matlab code to plot the following situations

m	k	c	ζ	ω_n	ω_d	x_o	v_o
10	10000	1				2.0	10.0
10	10000		.5			6.0	0.0
10	10000		.95			5.0	10.0
10	10000	1	1.0			5.0	10.0
10	10000	1	1.05			5.0	10.0

Find all the constants in equations 1.50, 1.63, 1.71

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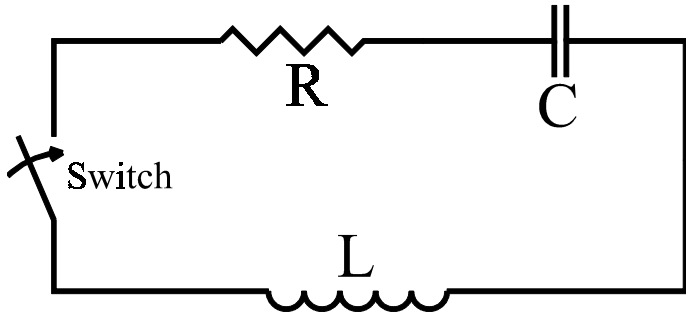


Figure 1.10: LRC Electrical Circuit

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Chapter 2

Forced Vibration

2.1 General Solution

In this section we look at the general solution to our second order differential equation. This section is developed with the idea of becoming a general reference for later sections. In these later sections specific cases, or geometries, will be investigated. These general cases, vary in how their differential equation is derived yet the form of the differential equation remains the same. Only the right hand side forcing function changes. The following section will address forcing functions which can be constructed from sines and cosines.

2.1.1 Undamped Systems

First we consider a non damped case, in which the differential equation is,

$$\ddot{x}(t) + \omega^2 x(t) = f(t) \tag{2.1}$$

Where $f(t)$ is a general function of time. We have already seen the homogeneous form of this equation. Now if we consider the nature of the differential equation we see this equation is in fact linear. This is due to the that taking the differential of a function is linear.

$$\frac{d}{dt}\left[y_1(t) + y_2(t)\right] = \frac{dy_1(t)}{dt} + \frac{dy_2(t)}{dt} \tag{2.2}$$

We can now consider our differential equation and consider the solution as the sum of two portions. The first of these two portions is the homogeneous solution, $x_h(t)$, which was already obtained in the first chapter. The second portion of the solution is a direct result of the right hand side of the differential equation and is called

the particular solution, $x_p(t)$. The complete solution to the differential equation is the sum of the homogeneous and particular solution. Why? If we write the total solution and place this into the differential equation we can answer the question, 'why'.

$$\ddot{x}_p(t) + \ddot{x}_h(t) + \omega^2 x_h(t) + \omega^2 x_p(t) = f(t) \tag{2.3}$$

However, we know the homogeneous form of the differential equation satisfies,

$$\ddot{x}_h(t) + \omega^2 x_h(t) = 0 \tag{2.4}$$

If we now subtract equation 2.4 from 2.3 the result will give,

$$\ddot{x}_p(t) + \omega^2 x_p(t) = f(t) \tag{2.5}$$

Since, the sum of both equations satisfy the total differential equation then the sum of both solution must be the solution. If we chose to ignore the homogeneous then we will be unable to satisfy our initial conditions.

Our goal now becomes the solution of the particular portion of this equation. To accomplish this we now turn to the method of undetermined coefficients. Luckily for our cases the right hand side function will at most be the sum of sines and cosines. If this is the case our equation becomes,

$$\ddot{x}_p(t) + \omega^2 x_p(t) = a_1 \sin \omega_f t + a_2 \cos \omega_f t \tag{2.6}$$

There are two cases which must be considered here. First, if $\omega_f \neq \omega$, meaning the forcing frequency is different that the natural frequency.

Non Resonant Forcing

For the case where $\omega_f \neq \omega$ our solution is of the form,

$$x_p(t) = A_1 \sin \omega_f t + A_2 \cos \omega_f t \tag{2.7}$$

Our goal at this point is to determine the coefficients A_1 and A_2 in terms of the known coefficients a_1 and a_2. We now take two derivatives of our solution to obtain,

$$\begin{aligned} \dot{x}_p(t) &= A_1 \omega_f \cos \omega_f t - A_2 \omega_f \sin \omega_f t \\ \ddot{x}_p(t) &= -A_1 \omega_f^2 \sin \omega_f t - A_2 \omega_f^2 \cos \omega_f t \end{aligned} \tag{2.8}$$

placing these equations back into the differential equation we obtain,

$$A_1 \left(\omega^2 - \omega_f^2\right) \sin \omega_f t + A_2 \left(\omega^2 - \omega_f^2\right) \cos \omega_f t = a_1 \sin \omega_f t + a_2 \cos \omega_f t \tag{2.9}$$

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Which gives a solution of

$$x_p(t) = A_1 \sin \omega_f t + A_2 \cos \omega_f t \tag{2.10}$$

$$\begin{aligned} A_1 &= \frac{a_1}{\omega^2 - \omega_f^2} \\ A_2 &= \frac{a_2}{\omega^2 - \omega_f^2} \end{aligned} \tag{2.11}$$

Hence the particular solution becomes,

$$\boxed{x_p(t) = \frac{a_1}{\omega^2 - \omega_f^2} \sin \omega_f t + \frac{a_2}{\omega^2 - \omega_f^2} \cos \omega_f t} \tag{2.12}$$

Now if we consider both the homogeneous and particular solution combined we get the solution as,

$$x(t) = C_1 \sin \omega t + C_2 \cos \omega t + \frac{a_1}{\omega^2 - \omega_f^2} \sin \omega_f t + \frac{a_2}{\omega^2 - \omega_f^2} \cos \omega_f t \tag{2.13}$$

Where the frequency ω is the natural frequency and C_1 and C_2 are constants to be determined by the initial conditions. We now seek these constants which satisfy the initial conditions.

$$\begin{aligned} x(0) &= C_2 + \frac{a_2}{\omega^2 - \omega_f^2} = x_0 \\ \dot{x}(0) &= C_1 \omega + \frac{a_1 \omega_f}{\omega^2 - \omega_f^2} = v_0 \end{aligned} \tag{2.14}$$

The general solution now becomes,

$$\boxed{\begin{aligned} x(t) = &\left(\frac{v_0}{\omega} - \frac{a_1 \omega_f}{\omega \left(\omega^2 - \omega_f^2 \right)} \right) \sin \omega t + \left(x_0 - \frac{a_2}{\omega^2 - \omega_f^2} \right) \cos \omega t \\ &+ \frac{a_1}{\omega^2 - \omega_f^2} \sin \omega_f t + \frac{a_2}{\omega^2 - \omega_f^2} \cos \omega_f t \end{aligned}} \tag{2.15}$$

Or through use of the phasor diagram we can rewrite this solution to,

$$\boxed{x(t) = \mathcal{A} \cos \left(\omega t - \phi_1 \right) + \mathcal{B} \cos \left(\omega_f t - \phi_2 \right)} \tag{2.16}$$

Where,

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$$\boxed{\begin{aligned}\mathcal{A} &= \sqrt{\left(x_0 + \frac{a_2}{-\omega^2 + {\omega_f}^2}\right)^2 + \frac{\left(v_0 + \frac{a_1\,\omega_f}{-\omega^2+{\omega_f}^2}\right)^2}{\omega^2}} \\ \phi_1 &= \frac{\omega\,\left(-a_2 + x_0\,\left(\omega^2 - {\omega_f}^2\right)\right)}{-\left(a_1\,\omega_f\right) + v_0\,\left(\omega^2 - {\omega_f}^2\right)} \\ \mathcal{B} &= \frac{\sqrt{a_1^2 + a_2^2}}{\omega^2 - \omega_f^2} \\ \phi_2 &= \tan^{-1}\left(\frac{a_1}{a_2}\right)\end{aligned}} \tag{2.17}$$

Case I

For a specific case we consider on cosine forcing as given in the following system,

$$\ddot{x}(t) + \omega^2 x(t) = f_0 \cos\omega_{dr} t \tag{2.18}$$

For this case the constants are

$$\begin{aligned} a_1 &= 0 \\ a_2 &= f_0 \end{aligned} \tag{2.19}$$

We get a solution,

$$\boxed{x(t) = \frac{v_0}{\omega}\sin\omega t + \left(x_0 - \frac{f_0}{\omega^2 - \omega_f^2}\right)\cos\omega t + \frac{f_0}{\omega^2 - \omega_f^2}\cos\omega_f t} \tag{2.20}$$

Case II

For a specific case we consider on cosine forcing as given in the following system,

$$\ddot{x}(t) + \omega^2 x(t) = f_0 \sin\omega_{dr} t \tag{2.21}$$

For this case the constants are

$$\begin{aligned} a_1 &= f_0 \\ a_2 &= 0 \end{aligned} \tag{2.22}$$

We get a solution,

$$\boxed{x(t) = \left(\frac{v_0}{\omega} - \frac{f_0\omega_f}{\omega\left(\omega^2 - \omega_f^2\right)}\right)\sin\omega t + x_0\cos\omega t + \frac{f_o}{\omega^2 - \omega_f^2}\sin\omega_f t} \tag{2.23}$$

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Forcing at Resonance

If we apply a forcing function at the same frequency as the resonate frequency of the system we a catastrophic situation, from the perspective of an engineer. If we return to our differential equation, with a bit of logical maneuvers, we can determine a solution. We still have the homogeneous solution $x_h(t)$, which is totally independent of any right hand side forcing function. As before we can subtract this portion out of the differential equation to obtain,

$$\ddot{x}_p(t) + \omega^2 x_p(t) = a_1 \sin \omega_f t + a_2 \cos \omega_f t \tag{2.24}$$

Now the frequency ω_f is at resonance. Considering the particular solution from before,

$$x_p(t) = A_1 \sin \omega_f t + A_2 \cos \omega_f t \tag{2.25}$$

We immediately see a problem with this assumed form of the solution. Since ω_f is ω placing this equation back into the differential equation satisfies **only** the homogeneous portion of the solution. In other words we can not get the left side of the equation to equal the right hand side.

Proof 2.1.1 *Taking the particular solution and setting $\omega_f = \omega$ we get*

$$\begin{aligned} x_p(t) &= A_1 \sin \omega_f t + A_2 \cos \omega_f t \\ \ddot{x}_p(t) &= -A_1 \omega^2 \sin \omega t - A_2 \omega^2 \cos \omega t \end{aligned} \tag{2.26}$$

If we multiply the first equation by ω^2 we get,

$$\begin{aligned} \omega^2 x_p(t) &= +A_1 \omega^2 \sin \omega_f t + A_2 \omega^2 \cos \omega_f t \\ \ddot{x}_p(t) &= -A_1 \omega^2 \sin \omega t - A_2 \omega^2 \cos \omega t \end{aligned} \tag{2.27}$$

Its now easy to see that is we added these two equations together we get zero. This makes sense due to the fact that this is the homogeneous solution and not the particular solution.

So what is the particular solution? What can we add to its own second derivative such that we get the right hand side forcing function? To this end we observe something interesting, if we take the sine and cosine multiplied by time we get,

$$x_p(t) = A_1 t \sin \omega t + A_2 t \cos \omega t \tag{2.28}$$

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If we take two derivatives to this equation we get,

$$\begin{aligned} \dot{x}_p(t) &= A_1 t\omega \cos\omega t + A_1 \sin\omega t + A_2 \cos\omega t - A_2 t\omega \sin\omega t \\ \ddot{x}_p(t) &= A_1 2\omega \cos\omega t - A_1 t\omega^2 \sin\omega t - A_2 t\omega^2 \cos\omega t - A_2 2\omega \sin\omega t \end{aligned} \tag{2.29}$$

Now we multiply $x_p(t)$ by ω^2 and place next to the acceleration term to see,

$$\begin{aligned} \omega^2 x_p(t) &= +A_1 t\omega^2 \sin\omega t + A_2 t\omega^2 \cos\omega t \\ \ddot{x}_p(t) &= -A_1 t\omega^2 \sin\omega t - A_2 t\omega^2 \cos\omega t + A_1 2\omega \cos\omega t - A_2 2\omega \sin\omega t \end{aligned} \tag{2.30}$$

Adding these two equations together and noticing that the result is in fact the differential equation we get,

$$+A_1 2\omega \cos\omega t - A_2 2\omega \sin\omega t = a_1 \sin\omega t + a_2 \cos\omega t \tag{2.31}$$

Therefore the particular solution becomes,

$$x_p(t) = \frac{a_2}{2\omega} t \sin\omega t - \frac{a_1}{2\omega} t \cos\omega t \tag{2.32}$$

The complete solution becomes

$$x(t) = C_1 \sin\omega t + C_2 \cos\omega t + \frac{a_2}{2\omega} t \sin\omega t - \frac{a_1}{2\omega} t \cos\omega t \tag{2.33}$$

2.2 Forced Response of Damped Systems

The differential equation for damped systems is

$$\ddot{x} + 2\zeta\omega\dot{x} + \omega^2 x = f_0 \cos\omega_{dr} t \tag{2.34}$$

First we seek the particular solution in the form of the right hand side and all of its derivatives.

$$x_p(t) = A_s \cos(t\,\omega_{dr}) + B_s \sin(t\,\omega_{dr}) \tag{2.35}$$

Then place this particular solution into the differential equation we get two equation and two unknowns

$$\begin{aligned} \omega^2\, B_s - 2\,\zeta\,\omega\, A_s\,\omega_{dr} - B_s\,{\omega_{dr}}^2 &= 0 \\ \omega^2\, A_s - f_0 + 2\,\zeta\,\omega\, B_s\,\omega_{dr} - A_s\,{\omega_{dr}}^2 &= 0 \end{aligned} \tag{2.36}$$

From these two equations we get,

$$\begin{aligned} A_s &= \frac{f_0\ \left(\omega^2 - {\omega_{dr}}^2\right)}{\omega^4 + 2\ \left(-1 + 2\,\zeta^2\right)\ \omega^2\,{\omega_{dr}}^2 + {\omega_{dr}}^4} \\ B_s &= \frac{2\,\zeta\,\omega\, f_0\,\omega_{dr}}{\omega^4 + 2\ \left(-1 + 2\,\zeta^2\right)\ \omega^2\,{\omega_{dr}}^2 + {\omega_{dr}}^4} \end{aligned} \tag{2.37}$$

Now comes the fun of placing these back into the equation.

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2.3 Phasors

However lets pause for a moment and investigate some phaser plots. Looking at this figure we associate the x-axis with $\sin$ and the y axis with $\cos$. This phasor plot can be understood better if we consider these vectors rotating with a constant speed of ω. In addition the y component of both vectors that's scribe out through time is the time function. Looking at the magnitude of the resultant vector we have,

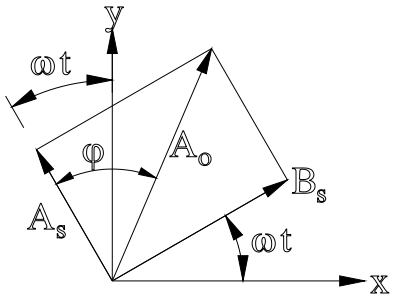


Figure 2.1: Phasor Vector Plots

$$A_0 = \sqrt{A_s^2 + B_s^2} \tag{2.38}$$

In addition the phase angle is,

$$\phi = \tan^{-1}\left(\frac{B_s}{A_s}\right) \tag{2.39}$$

Now we can find the angle between the y axis and vector A_0. This angle will reference the vector to the cosine. Finding the angle between the vector A_0 and the x axis will reference the vector to the sine. Therefore the resultant vector can be written either way,

$$\begin{aligned} A_s \cos\omega t + B_s \sin\omega t &= A_0 \cos\left(\omega t - \phi\right) \\ A_s \cos\omega t + B_s \sin\omega t &= A_0 \sin\left(\omega t + \frac{\pi}{2} - \phi\right) \end{aligned} \tag{2.40}$$

If we look back at equation 2.37 we see the magnitude of the vector as,

$$A_0 = \sqrt{\frac{{f_0}^2}{\omega^4 + 2\left(-1 + 2\,\zeta^2\right)\omega^2\,{\omega_{dr}}^2 + {\omega_{dr}}^4}} \tag{2.41}$$

The phase angle for this case is,

$$\phi = \tan^{-1}\frac{2\,\zeta\,\omega\,\omega_{dr}}{\omega^2 - {\omega_{dr}}^2} \tag{2.42}$$

This gives a solution of

$$x_p\left(t\right) = \sqrt{\frac{{f_0}^2}{\omega^4 + 2\left(-1 + 2\,\zeta^2\right)\omega^2\,{\omega_{dr}}^2 + {\omega_{dr}}^4}}\cos\left(\omega_{dr}t - \tan^{-1}\left(\frac{2\,\zeta\,\omega\,\omega_{dr}}{\omega^2 - {\omega_{dr}}^2}\right)\right) \tag{2.43}$$

Which is equation (T2.28).

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2.4 Base Excitation

The differential equation of motion is;

$$\ddot{x} + 2\zeta\omega\dot{x} + \omega^2 x = 2\zeta\omega\omega_b \mathbf{Y}\cos\omega_b t + \omega^2 \mathbf{Y}\sin\omega_b t \tag{2.44}$$

If we start by looking at the particular solution, we first look at the right hand side to construct the form of the solution. We take the right hand side and all of the derivatives. Hence the particular solution will look like,

$$x_p(t) = A_s\cos\omega_b t + B_s\sin\omega_b t \tag{2.45}$$

If we now

$$\begin{aligned} \dot{x}_p(t) &= -\left(\sin(t\,\omega_b)\,A_s\,\omega_b\right) + \cos(t\,\omega_b)\,B_s\,\omega_b \\ \ddot{x}_p(t) &= -\left(\cos(t\,\omega_b)\,A_s\,{\omega_b}^2\right) - \sin(t\,\omega_b)\,B_s\,{\omega_b}^2 \end{aligned} \tag{2.46}$$

Placing these equations into the differential equation we get two equations and two unknowns,

$$\begin{aligned} -\left(Y\,\omega^2\right) + \omega^2\,B_s - 2\,\zeta\,\omega\,A_s\,\omega_b - B_s\,{\omega_b}^2 = 0 \\ \omega^2\,A_s - 2\,Y\,\zeta\,\omega\,\omega_b + 2\,\zeta\,\omega\,B_s\,\omega_b - A_s\,{\omega_b}^2 = 0 \end{aligned} \tag{2.47}$$

From these two equations we get a solution of,

$$A_s = \frac{-2\,Y\,\zeta\,\omega\,{\omega_b}^3}{\omega^4 - 2\,\omega^2\,{\omega_b}^2 + 4\,\zeta^2\,\omega^2\,{\omega_b}^2 + {\omega_b}^4} \tag{2.48}$$

$$B_s = -\frac{-\left(Y\,\omega^4\right) + Y\,\omega^2\,{\omega_b}^2 - 4\,Y\,\zeta^2\,\omega^2\,{\omega_b}^2}{\omega^4 - 2\,\omega^2\,{\omega_b}^2 + 4\,\zeta^2\,\omega^2\,{\omega_b}^2 + {\omega_b}^4} \tag{2.49}$$

Now if we take this solution back to the phasor diagram we get,

$$A_0 = \sqrt{\frac{Y^2\,\omega^2\,\left(\omega^2 + 4\,\zeta^2\,{\omega_b}^2\right)}{\omega^4 + 2\,\left(-1 + 2\,\zeta^2\right)\,\omega^2\,{\omega_b}^2 + {\omega_b}^4}} \tag{2.50}$$

The phase angle is,

$$\phi = \tan^{-1}\left[\frac{-\left(\omega\,\left(\omega^2 + \left(-1 + 4\,\zeta^2\right)\,{\omega_b}^2\right)\right)}{2\,\zeta\,{\omega_b}^3}\right] \tag{2.51}$$

Combining all this crap we get a solution of,

$$\begin{aligned} x_p(t) = \sqrt{\frac{Y^2\,\omega^2\,\left(\omega^2 + 4\,\zeta^2\,{\omega_b}^2\right)}{\omega^4 + 2\,\left(-1 + 2\,\zeta^2\right)\,\omega^2\,{\omega_b}^2 + {\omega_b}^4}}\times \\ \cos\left(\omega_b t - \tan^{-1}\left[\frac{-\left(\omega\,\left(\omega^2 + \left(-1 + 4\,\zeta^2\right)\,{\omega_b}^2\right)\right)}{2\,\zeta\,{\omega_b}^3}\right]\right) \end{aligned} \tag{2.52}$$

If we are interested in the magnitude

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2.5 Rotating Unbalance

A very common dynamic situation encountered is that of a rotating unbalanced machine. This situation is often seen and felt in an automobile, either in its main engine or in the wheels of the auto. Also this can be seen when a washing machine's load is slightly off center due to the clothes being off to one side of the drum. This off center load will cause the washing machine to jump around all over the floor like a small cat tossed on a bed of hot coals, please note I have not and will not do this nor have I any knowledge or at least can't remember anyone that has ever done this. This phenomena occurs due to a machines center of gravity being slight offset for its axis of rotation. A model of this situation is shown in figure 2.2. In this situation the rotating mass, m_0, is a distance of e away from its own center of rotation. However, this mass center moves up and down as the machine itself moves. If we stop and draw a freebody diagram of this situation we see figure 2.3. Notice in this free body diagram the coordinate system, *and for that matter all coordinate systems*, must be referenced to a fixed location. If we wish to determine the force F_x we must use the free body diagram on the right. From this free body diagram the right we see the distance from

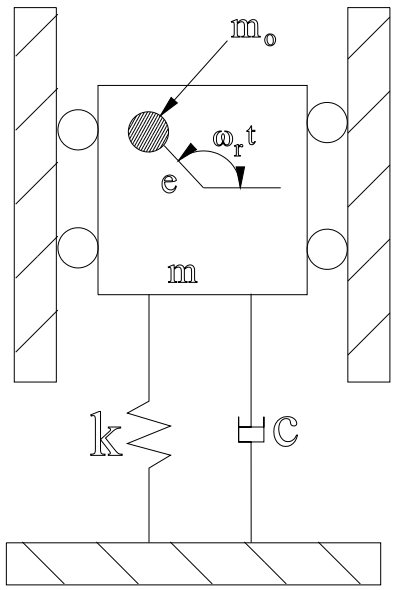


Figure 2.2: Rotating Mass System

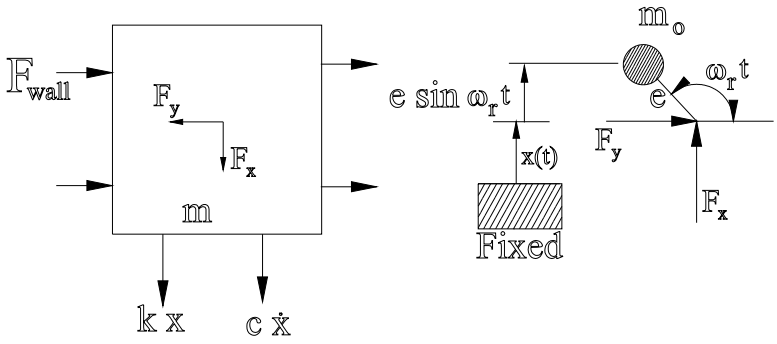


Figure 2.3: Free Body Diagram

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the fixed location to the mass m_0 is;

$$x_{m_0}(t) = x(t) + e\sin\omega_r t \tag{2.53}$$

From this distance we can take two derivatives to find the acceleration of this mass. Once we do this we use Newton's second law to this system to get,

$$F_x = m_0\frac{d^2}{dt^2}\left[x(t) + e\sin\omega_r t\right] = m_0\ddot{x}(t) - m_0 e\omega_r^2\sin\omega_r t \tag{2.54}$$

Now we can take this F_x from one free body diagram to the other. Now looking at the free body diagram on the left of this figure and noticing the force of the rollers on the wall, F_{wall} cancels the F_y we concentrate only on the vertical motion. Writing Newton's second law in this direction we get,

$$-kx - c\dot{x} - F_x = m\ddot{x} \tag{2.55}$$

Inserting the value of this force we get,

$$-kx - c\dot{x} - m_0\ddot{x}(t) + m_0 e\omega_r^2\sin\omega_r t = m\ddot{x} \tag{2.56}$$

Rearranging we get

$$(m + m_0)\,\ddot{x} + c\dot{x} + kx = m_0 e\omega_r^2\sin\omega_r t \tag{2.57}$$

The total mass of the machine is the sum of the mass of the housing, m, and the mass of the rotation portion, m_0. We shall call the total mass of the machine M for convenience. Therefore, the equation of motion becomes,

$$M\ddot{x} + c\dot{x} + kx = m_0 e\omega_r^2\sin\omega_r t \tag{2.58}$$

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Chapter 3

Two Degree of Freedom

In this chapter we will go through the complete development of a two degree of freedom system. This will include the development of the 2 governing equations which result from the $\sum \overrightarrow{F} = m\overrightarrow{a}$. From these equations we will see how and why eigenvalues problems are so important.

3.1 Problem Development

In figure 1 the two masses are frozen in time, as if we took a photo of the motion. In this figure mass 2 appears as if it has displaced further to the right than mass 1.

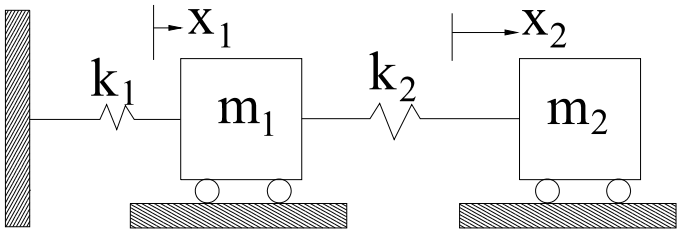


Figure 3.1: Physical Problem

We shall use this assumption to derive the differential equations of motion. In equation form this deflection situation can be written as,

$$x_2 > x_1. \tag{3.1}$$

If this is the case then spring 2 is stretched and is pulling mass 1 towards mass 2 and vice versa. This situation is shown in figure 2, the Free Body Diagram. From

this figure we sum the forces on mass 1 to obtain

$$\sum F = ma = k_2(x_2 - x_1) - k_1 x_1 = m_1 \ddot{x}_1 \tag{3.2}$$

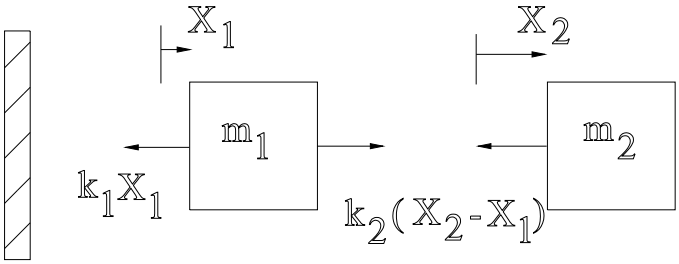


Figure 3.2: Free Body Diagram

Now if we do the same for the second mass we obtain;

$$\sum F = ma = -k_2(x_2 - x_1) = m_2 \ddot{x}_2 \tag{3.3}$$

One note should be stated at this point, if we had assumed that $x_2 < x_1$ the final outcome of the derivation would be unchanged, this proof is left up to the reader. The result of this would give two equations,

$$\begin{aligned} m_1 \ddot{x}_1 - k_2 x_2 + k_2 x_1 + k_1 x_1 = 0 \\ m_2 \ddot{x}_2 + k_2 x_2 - k_2 x_1 = 0 \end{aligned} \tag{3.4}$$

In matrix form these equations become,

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3.5}$$

Where

$$\begin{aligned} k_{11} &= k_1 + k_2 \\ k_{12} &= -k_2 \\ k_{22} &= k_2 \end{aligned} \tag{3.6}$$

The equation has be placed in general form from this point on. This was done in order to consider the most general two degree of freedom case. If we look at these two equations long enough we see the coupling between them. For example, if k_2 is zero the the off diagonal terms in the matrices are all zero and there are two

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independent equations. However, this is not the case. In fact if we write out each equation we see the non-diagonal term,

$$\begin{aligned} m_1\ddot{x}_1 + k_{11}x_1 + \underbrace{k_{12}x_2} &= 0 \\ m_2\ddot{x}_2 + k_{22}x_2 + \underbrace{k_{12}x_1} &= 0 \end{aligned} \tag{3.7}$$

The brace is under the term which connects the 2 differential equations together, the off diagonal terms in the matrix. As we shall see later forcing the mass and stiffness matrix into a diagonal form will break the coupling between the equations.

We now seek a solution to these two simultaneous equations.

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3.8}$$

Since these equations are homogeneous any solution and a scalar multiple is also a solution. I.E. if $x_1(t)$ and $x_2(t)$ is a solution so to is $\mathcal{C}x_1(t)$ and $\mathcal{C}x_2(t)$ where $\mathcal{C}$ is an arbitrary constant.

We now make an assumption which we shall return later to prove correct. We assume the displacement of mass 1 is a scalar constant times a function, and the displacement of mass 2 is a different scalar constant time this same function. This type of motion is called *synchronous*. In fact, linear multidegree of freedom systems always admit *synchronous* motion.

$$\begin{aligned} x_1(t) &= A_1\, f(t) \\ x_2(t) &= A_2\, f(t) \end{aligned} \tag{3.9}$$

We shall return later to prove this assumption is correct. With this assumption our equations can be written as,

$$\begin{aligned} m_1A_1\ddot{f}(t) + [k_{11}A_1 + k_{12}A_2]\, f(t) &= 0 \\ m_2A_2\ddot{f}(t) + [k_{12}A_1 + k_{22}A_2]\, f(t) &= 0 \end{aligned} \tag{3.10}$$

In this equation terms, m_1, m_2, k_{11}, k_{12}, k_{22}, A_1 and A_2 are all real constants. Rewriting these two equations we obtain,

$$-\frac{\ddot{f}(t)}{f(t)} = \frac{[k_{11}A_1 + k_{12}A_2]}{m_1A_1} = \frac{[k_{12}A_1 + k_{22}A_2]}{m_2A_2} = \lambda \tag{3.11}$$

Since A_1, A_2, k_{11}, k_{12}, k_{22}, m_1 and m_2 are all constants, so too is λ. The equation can now be rewritten and solved for $f(t)$.

$$\ddot{f}(t) + \lambda\, f(t) = 0 \tag{3.12}$$

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This equation should look familiar, in chapter 1 we covered the solution to an equation exactly like this one. The solution to this equation can take several forms. In this case we shall use the solution in the form

$$f(t) = C\cos(\omega t + \phi) \tag{3.13}$$

Using this equation and its second derivatives we get,

$$-C\omega^2\cos(\omega t + \phi) + \lambda\, C\cos(\omega t + \phi) = 0 \tag{3.14}$$

Now since $C\cos(\omega t + \phi)$ is not always zero for all time we divide it out to obtain.

$$\omega^2 = \lambda \tag{3.15}$$

Now we rewrite equation 3.11 as,

$$\frac{[k_{11}A_1 + k_{12}A_2]}{m_1A_1} = \frac{[k_{12}A_1 + k_{22}A_2]}{m_2A_2} = \omega^2 \tag{3.16}$$

Since there are actually two equations represented in this case, we split them apart to get,

$$\begin{aligned}[k_{11}A_1 + k_{12}A_2] - m_1A_1\omega^2 = 0\\ [k_{12}A_1 + k_{22}A_2] - m_2A_2\omega^2 = 0\end{aligned} \tag{3.17}$$

Or in matrix form we get,

$$\begin{bmatrix}k_{11} & k_{12}\\ k_{12} & k_{22}\end{bmatrix}\begin{Bmatrix}A_1\\ A_2\end{Bmatrix} - \begin{bmatrix}m_1\omega^2 & 0\\ 0 & m_2\omega^2\end{bmatrix}\begin{Bmatrix}A_1\\ A_2\end{Bmatrix} = \begin{Bmatrix}0\\ 0\end{Bmatrix} \tag{3.18}$$

Or combining terms into one matrix we get,

$$\begin{bmatrix}k_{11} - m_1\omega^2 & k_{12}\\ k_{12} & k_{22} - m_2\omega^2\end{bmatrix}\begin{Bmatrix}A_1\\ A_2\end{Bmatrix} = \begin{Bmatrix}0\\ 0\end{Bmatrix} \tag{3.19}$$

Remembering that A_1 and A_2 are the amplitudes of the vibration of mass 1 and 2 respectively, we conclude that in order to have a non-trivial solution neither constant should be zero. Using Cramers rule to solve for A_1 we get,

$$A_1 = \frac{\begin{vmatrix}0 & k_{12}\\ 0 & k_{22} - m_2\omega^2\end{vmatrix}}{\begin{vmatrix}k_{11} - m_1\omega^2 & k_{12}\\ k_{12} & k_{22} - m_2\omega^2\end{vmatrix}} = \frac{0}{\begin{vmatrix}k_{11} - m_1\omega^2 & k_{12}\\ k_{12} & k_{22} - m_2\omega^2\end{vmatrix}} \tag{3.20}$$

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Now the only way we can have $A_1 \neq 0$ in this equation is if the equation is indeterminate. Or in mathematical terms this equation becomes,

$$A_1 = \frac{0}{\begin{vmatrix} k_{11} - m_1\omega^2 & k_{12} \\ k_{12} & k_{22} - m_2\omega^2 \end{vmatrix}} = \frac{0}{0} \neq 0 \tag{3.21}$$

Therefore in order to obtain non trivial solutions for both A_1 and A_2 we must render the matrix singular by forcing its determinate to zero.

$$\begin{vmatrix} k_{11} - m_1\omega^2 & k_{12} \\ k_{12} & k_{22} - m_2\omega^2 \end{vmatrix} = 0 \tag{3.22}$$

The characteristic equation from this determinate becomes,

$$\left(k_{11} - m_1\omega_i^2\right)\left(k_{22} - m_2\omega_i^2\right) - k_{12}^2 = 0 \tag{3.23}$$

Rearranging,

$$m_1 m_2 \omega_i^4 - \left(k_{11}m_2 + m_1 k_{22}\right)\omega_i^2 + k_{11}k_{22} - k_{12}^2 = 0 \tag{3.24}$$

Using the quadratic formula to find the two roots of this equation we obtain,

$$\omega_i^2 = \frac{k_{11}m_2 + m_1 k_{22}}{2m_1 m_2} \pm \frac{\sqrt{\left(k_{11}m_2 + m_1 k_{22}\right)^2 - 4m_1 m_2\left(k_{11}k_{22} - k_{12}^2\right)}}{2m_1 m_2} \tag{3.25}$$

Example 3.1.1

Lets take the case where $k_1 = k_2 = 2$ and $m_1 = m_2 = 1$. The resulting determinate would give,

$$\begin{vmatrix} 4 - \omega^2 & -2 \\ -2 & 2 - \omega^2 \end{vmatrix} = 0 \tag{3.26}$$

This equation yields the following polynomial in ω^2,

$$\left(4 - \omega^2\right)\left(2 - \omega^2\right) - 4 = 0 \tag{3.27}$$

Or

$$\omega^4 - 6\omega^2 + 4 = 0 \tag{3.28}$$

Which gives a solution of

$$\begin{aligned} \omega_1^2 &= 3 - \sqrt{5} \\ \omega_2^2 &= 3 + \sqrt{5} \end{aligned} \tag{3.29}$$

This Completes The Example

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Example 3.1.2

Lets take the case where $k_1 = k_2 = 2$ and $m_1 = m_2 = 1$. The resulting determinate would give,

$$\begin{vmatrix} 4-\omega^2 & -2 \\ -2 & 2-\omega^2 \end{vmatrix} = 0 \tag{3.30}$$

This equation yields the following polynomial in ω^2,

$$\left(4-\omega^2\right)\left(2-\omega^2\right) - 4 = 0 \tag{3.31}$$

Or

$$\omega^4 - 6\omega^2 + 4 = 0 \tag{3.32}$$

Which gives a solution of

$$\begin{aligned} \omega_1^2 &= 3-\sqrt{5} \\ \omega_2^2 &= 3+\sqrt{5} \end{aligned} \tag{3.33}$$

This Completes The Example

Now we return to equation 3.13 noticing there exist two distinct frequencies which apply to this equation. Each displacement x_1 and x_2 are scalars time the same function $f(t)$. In this case we have two different functions which satisfy the differential equations, the functions are different only due to the change in frequency. We make use of linear superposition to obtain the complete solution.

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = \mathcal{A}_1 \begin{Bmatrix} a_{11} \\ a_{21} \end{Bmatrix} \cos\left(\omega_1 t + \phi\right) + \mathcal{A}_2 \begin{Bmatrix} a_{12} \\ a_{22} \end{Bmatrix} \cos\left(\omega_2 t + \phi\right) \tag{3.34}$$

Where $\mathcal{A}_1$, $\mathcal{A}_2$, a_{11}, a_{21}, a_{12} and a_{22} are all constants. Taking two derivatives of the solution we obtain,

$$\begin{Bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{Bmatrix} = -\mathcal{A}_1\omega_1^2 \begin{Bmatrix} a_{11} \\ a_{21} \end{Bmatrix} \cos\left(\omega_1 t + \phi\right) - \mathcal{A}_2\omega_2^2 \begin{Bmatrix} a_{12} \\ a_{22} \end{Bmatrix} \cos\left(\omega_2 t + \phi\right) \tag{3.35}$$

Since there are two solutions and the differential equations are linear we can consider each solution separately as we shall do in the following. First we consider the

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solution associated with ω_1. Taking this solution

$$\begin{aligned} x_{11}(t) &= a_{11}\cos(\omega_1 t + \phi) \\ x_{21}(t) &= a_{21}\cos(\omega_1 t + \phi) \\ \ddot{x}_{11}(t) &= -a_{11}\omega_1^2\cos(\omega_1 t + \phi) \\ \ddot{x}_{21}(t) &= -a_{21}\omega_1^2\cos(\omega_1 t + \phi) \end{aligned} \tag{3.36}$$

and placing into the differential equations we obtain,

$$\cos(\omega_1 t + \phi)\left[\begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix}\begin{Bmatrix} a_{11} \\ a_{21} \end{Bmatrix} - \begin{bmatrix} m_1\omega_1^2 & 0 \\ 0 & m_2\omega_1^2 \end{bmatrix}\begin{Bmatrix} a_{11} \\ a_{21} \end{Bmatrix}\right] = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3.37}$$

The term $\cos(\omega_1 t + \phi)$ is not zero for all time, therefore we can divide this term out to obtain,

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix}\begin{Bmatrix} a_{11} \\ a_{21} \end{Bmatrix} - \begin{bmatrix} m_1\omega_1^2 & 0 \\ 0 & m_2\omega_1^2 \end{bmatrix}\begin{Bmatrix} a_{11} \\ a_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3.38}$$

Or

$$\begin{bmatrix} k_{11} - m_1\omega_1^2 & k_{12} \\ k_{12} & k_{22} - m_2\omega_1^2 \end{bmatrix}\begin{Bmatrix} a_{11} \\ a_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3.39}$$

Example 3.1.3

Let us now look at our example where $k_1 = k_2 = 2$ and $m_1 = m_2 = 1$. Remembering that our eigenvalues were $\omega_1^2 = 3 - \sqrt{5}$ and $\omega_2^2 = 3 + \sqrt{5}$. For ω_1^2 we get,

$$\begin{bmatrix} 1+\sqrt{5} & -2 \\ -2 & -1+\sqrt{5} \end{bmatrix}\begin{Bmatrix} a_{11} \\ a_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3.40}$$

Likewise for the second frequency we get;

$$\begin{bmatrix} k_1 + k_2 - m_1\omega_2^2 & -k_2 \\ -k_2 & k_2 - m_2\omega_2^2 \end{bmatrix}\begin{Bmatrix} a_{12} \\ a_{22} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3.41}$$

Now if we try to solve for a_{11} and a_{21} what we find out about these two equations are, they are linearly dependent. I.E. the matrix is singular. This is a result of our attempt to force this matrix's determinate to zero, or we made the matrix singular. Therefore, we only have one equation and two unknowns. The only thing we can do at this point is to determine the ratio,

$$\frac{a_{21}}{a_{11}} = \frac{1+\sqrt{5}}{2} \tag{3.42}$$

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OR

$$\begin{Bmatrix} a_{11} \\ a_{21} \end{Bmatrix} = \begin{Bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1.61803 \end{Bmatrix} \tag{3.43}$$

If we now consider the second frequency of $\omega_2^2 = 3 + \sqrt{5}$ *we get,*

$$\begin{bmatrix} 1-\sqrt{5} & -2 \\ -2 & -1-\sqrt{5} \end{bmatrix} \begin{Bmatrix} a_{12} \\ a_{22} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3.44}$$

Again, we forced this matrix's determinate to zero by our choice of ω_2^2*. As before we have one equation and two unknowns.*

$$\begin{Bmatrix} a_{12} \\ a_{22} \end{Bmatrix} = \begin{Bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{Bmatrix} = \begin{Bmatrix} 1 \\ -0.618034 \end{Bmatrix} \tag{3.45}$$

With this our solution to the two differential equations becomes;

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = \mathcal{A}_1 \begin{Bmatrix} 1 \\ 1.61803 \end{Bmatrix} \cos(\omega_1 t + \phi) + \mathcal{A}_2 \begin{Bmatrix} 1 \\ -0.618034 \end{Bmatrix} \cos(\omega_2 t + \phi) \tag{3.46}$$

Looking at equations 3.43, 3.45 we see these vectors are the eigenvectors.

This Completes The Example

3.2 Eigenvectors

In this section we will examine some of the properties of the eigenvectors. First we notice the solution in equation 3.46 can be multiplied by any constant and the result will also satisfy the differential equation. Therefore, an eigenvector can be multiplied by an arbitrary constant. The result of this multiplication changes nothing. In other words if

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = \mathcal{A}_1 \begin{Bmatrix} 1 \\ 1.61803 \end{Bmatrix} \cos(\omega_1 t + \phi) + \mathcal{A}_2 \begin{Bmatrix} 1 \\ -0.618034 \end{Bmatrix} \cos(\omega_2 t + \phi) \tag{3.47}$$

satisfies the differential equation, so to does

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = \mathcal{A}_1 \begin{Bmatrix} 1 \times \mathcal{C} \\ 1.61803 \times \mathcal{C} \end{Bmatrix} \cos(\omega_1 t + \phi) + \mathcal{A}_2 \begin{Bmatrix} 1 \times \mathcal{D} \\ -0.618034 \times \mathcal{D} \end{Bmatrix} \cos(\omega_2 t + \phi) \tag{3.48}$$

Where $\mathcal{D}$ and $\mathcal{C}$ are arbitrary constants. Proof that this solution also satisfies the differential equation is left up to the reader. However, this observation brings up an

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interesting question; Does the actual values listed in each eigenvector mean anything or does only the ratio of one entry to the other mean something. Besides, the eigenvector was formed from one equation and two unknowns remember equations 3.41 to 3.45. In other words the solution to

$$\begin{bmatrix} 1+\sqrt{5} & -2 \\ -2 & -1+\sqrt{5} \end{bmatrix} \begin{Bmatrix} a_{11} \\ a_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3.49}$$

is

$$\begin{Bmatrix} a_{11} \\ a_{21} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1.61803 \end{Bmatrix} \tag{3.50}$$

However, another solution is

$$\begin{Bmatrix} a_{11} \\ a_{21} \end{Bmatrix} = \begin{Bmatrix} 0.525732 \\ 0.85065 \end{Bmatrix} \tag{3.51}$$

So how does equation 3.51 come from 3.50 or is its selection arbitrary? The answer to this question is no, its not arbitrary. The eigenvector is a vector that has a length associated with it. The eigenvector from equation 3.50 has a length of;

$$\mathcal{L} = \sqrt{1^2 + 1.61803^2} = 1.90211 \tag{3.52}$$

Now if divide each entry in eigenvector from equation 3.50 by the total length of the vector we get,

$$\begin{Bmatrix} a_{11} \\ a_{21} \end{Bmatrix} = \begin{Bmatrix} 1/1.90211 \\ 1.61803/1.90211 \end{Bmatrix} = \begin{Bmatrix} 0.525732 \\ 0.85065 \end{Bmatrix} \tag{3.53}$$

This means the length of the eigenvector in equation 3.51 is of unit length.

$$\mathcal{L} = \sqrt{0.525732^2 + 0.85065^2} = 1.0 \tag{3.54}$$

These are termed normalized eigenvectors. Normalizing both eigenvectors we obtain the solution as;

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = \mathcal{Z}_1 \begin{Bmatrix} 0.525732 \\ 0.85065 \end{Bmatrix} \cos(\omega_1 t + \phi) + \mathcal{Z}_2 \begin{Bmatrix} -0.85065 \\ 0.525731 \end{Bmatrix} \cos(\omega_2 t + \phi) \tag{3.55}$$

Or we could write this equation in matrix form as;

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = \underbrace{\begin{bmatrix} 0.525732 & -0.85065 \\ 0.85065 & 0.525731 \end{bmatrix}} \begin{Bmatrix} \mathcal{Z}_1 \cos(\omega_1 t + \phi) \\ \mathcal{Z}_2 \cos(\omega_2 t + \phi) \end{Bmatrix} \tag{3.56}$$

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The portion of the equation above is the matrix that MATLAB sends back from its `eig()` command, see page 106 of the text.

Now let's compare this matrix to the rotation matrix on page 102 of the text book. The rotation matrix on page 91 and 102 is;

$$[\mathcal{V}] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 0.525732 & -0.85065 \\ 0.85065 & 0.525731 \end{bmatrix} \tag{3.57}$$

From this equation we find $\theta = 1.0173$. What does this mean? Are the Eigenvectors an actual rotation of coordinates? The answer is, They are! In addition because we made each eigenvector a length of unity. This has an interesting effect. By doing this we can make a direct correlation between the eigenvector matrix $\mathcal{V}$ and the rotation matrix. One interesting thing to note is, the transpose of the rotation matrix (normalized eigenvector matrix $\mathcal{V}$) is its own inverse. in mathematical terms this is;

$$\begin{bmatrix} 0.525732 & -0.85065 \\ 0.85065 & 0.525731 \end{bmatrix}^{-1} = \begin{bmatrix} 0.525732 & 0.85065 \\ -0.85065 & 0.525731 \end{bmatrix} \tag{3.58}$$

This is left to the reader to prove.

Now if we decide to use this eigenvector matrix to rotate the general matrix, like that done on page 102 we see something very interesting. In our example we get,

$$\begin{bmatrix} 0.5257 & 0.8507 \\ -0.8507 & 0.5257 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 0.5257 & -.8507 \\ 0.8507 & 0.5257 \end{bmatrix} = \\ \begin{bmatrix} 0.7639 & 0.0000 \\ 0.0000 & 5.2361 \end{bmatrix} = \begin{bmatrix} 3-\sqrt{5} & 0 \\ 0 & 3+\sqrt{5} \end{bmatrix} = \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} \tag{3.59}$$

In other words if $\mathcal{V}$ is our normalized eigenvector matrix and $\mathcal{K}$ is the matrix which produced the eigenvalues then,

$$[\mathcal{V}]^{-1}[\mathcal{K}][\mathcal{V}] = [\mathcal{D}] \tag{3.60}$$

Where $[\mathcal{D}]$ is a diagonal matrix with the eigenvalues on the diagonals. In MATLAB the function `[V,D]=eig(A)` gives these two matrices back. Here $[\mathcal{V}]$ is the `[V]` from MATLAB and $[\mathcal{D}]$ is the [D] from MATLAB. Type `help eig` while in MATLAB to get a complete description of the command

There is another way of obtaining the same result. If we look at equation 3.38 which can be rewritten as,

$$\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} a_{1i} \\ a_{2i} \end{Bmatrix} = \begin{bmatrix} m_1\omega_i^2 & 0 \\ 0 & m_2\omega_i^2 \end{bmatrix} \begin{Bmatrix} a_{1i} \\ a_{2i} \end{Bmatrix} \tag{3.61}$$

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Notice this equation applies to both eigenvalue eigenvector combinations as indicated by the subscript i. If we spend some time thoughtfully manipulating this equation, as we shall do here, we will obtain the same result as equation 3.58 and 3.59. First lets remove the ω_i from the matrix to obtain,

$$\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} a_{1i} \\ a_{2i} \end{Bmatrix} = \omega_i^2 \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} a_{1i} \\ a_{2i} \end{Bmatrix} \tag{3.62}$$

Now we realize that there are actually two equations here. It would seem natural to try to turn the two equations into one equation in some manner. This next step will be assigned as a homework problem, but the equation above and the equation to follow are actually the same equation,

$$\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} \tag{3.63}$$

NOW IF AND ONLY IF BOTH MASSES m_1 and m_2 ARE OF A VALUE OF ONE (1) can we do the following. We shall address the general case of arbitrary masses shortly.

$$\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} \tag{3.64}$$

OR

$$\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} \tag{3.65}$$

Of course the next step is to premultiply both sides with the inverse of the eigenvector matrix. This gives,

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} \tag{3.66}$$

Now, this equation is the exact same as that of equation 3.59 and 3.60. One important thing to note in this derivation, *we never normalized the eigenvectors here*. In fact the only thing that results for normalizing our eigenvectors is that the inverse of the eigenvector matrix is the same as its transpose. As a result, normalizing the eigenvectors we can draw a corollary between rotation of coordinates and the eigenvalue problem in general. This is something we shall look into further in the next section.

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3.2.1 Functional Interpretation of the Eigenvalue problem

In matrix terms the eigenvalue problem could be written as,

$$[K]\{x\} = \lambda\{x\} \tag{3.67}$$

This general statement of the eigenvalue problem is not confined to a 2x2. However, we shall confine ourselves to this 2x2 problem for instructional purposes. If we premultiply equation 3.67 by $\{x\}^T$ we obtain,

$$\{x\}^T[K]\{x\} = \lambda\{x\}^T\{x\} \tag{3.68}$$

As before, we can normalize the vector $\{x\}$. equation 3.68 by $\{x\}^T$ we obtain,

$$\{x\}^T\{x\} = 1 \tag{3.69}$$

In other words we just made this a vector of length 1. Hence our equation now becomes,

$$\{x\}^T[K]\{x\} = \lambda \tag{3.70}$$

If we now confine ourselves to a 2x2 where we have a symmetric matrix we can write this equation as,

$$\begin{Bmatrix} x_1 & x_2 \end{Bmatrix} \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \lambda \tag{3.71}$$

Or multiplying out this equation we obtain,

$$\frac{1}{\lambda}\left(k_{11}x_1^2 + 2k_{12}x_1x_2 + k_{22}x_2^2\right) = 1 \tag{3.72}$$

This is an equation of a conic section, remember from Calculus? From Calculus we had the general form of a conic section as,

$$Ax_1^2 + Bx_1x_2 + Cx_2^2 + Dx_1 + Ex_2 + F = 0 \tag{3.73}$$

From which there was a Theorem which stated the following

- If $B^2 - 4AC = 0$ the equation is parabolic or, 2 parallel line or, is imaginary.
- If $B^2 - 4AC < 0$ the equation is elliptic or, circle or, one point or, is imaginary.
- If $B^2 - 4AC > 0$ the equation is hyperbolic or, 2 intersecting lines.

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If we pause for a moment and go all the way back to equation 3.5 and forcing the masses to each be one then we have the same problem which can be restated as,

$$\{x_1 \quad x_2\} \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \lambda \tag{3.74}$$

Or

$$\begin{aligned} A &= k_1 + k_2 \\ B &= -k_2 \\ C &= k_2 \end{aligned} \tag{3.75}$$

Hence

$$B^2 - 4AC = k_2\left(-4k_1 - 3k_2\right) = -k_2\left(4k_1 + 3k_2\right) < 0 \tag{3.76}$$

Hence, for vibration of systems we get an equation of a circle or an ellipse. This is always the case, in fact for all vibrational systems which are connected to a fixed location, say the earth, we always get an equation of a circle or an ellipse. This equation of actually corresponds to a system which gives all positive eigenvalues. This makes sense, have you ever seen a system vibrating at a negative frequency, what ever that means.

Now lets take our specific case and look at this equation of an ellipse. Our example was a case where $k_1 = k_2 = 2$ and $m_1 = m_2 = 1$. Remembering that our eigenvalues were $\omega_1^2 = 3 - \sqrt{5}$ and $\omega_2^2 = 3 + \sqrt{5}$ we develop up an actual equation for our ellipse. Our equation becomes,

$$f(x, y) = 4x^2 - 4xy + 2y^2 - \lambda \tag{3.77}$$

Note: the λ is merely a constant which raises or lowers the function $f(x, y)$, therefore its value in inconsequential to the shape of the contours, only to their value. Where we converted from x_1 to x and x_2 to y. If we contour this function, as shown in figure 3.3, we see a series of elliptic contours which are centered at the origin. Notice on this plot at the center I placed both the x-y coordinate axis as well as two other vectors. Upon closer examination you should see that these other vectors are actually the eigenvectors. The $e1$ corresponds to first eigenvector while the second vector $e2$ corresponds to the second eigenvector.

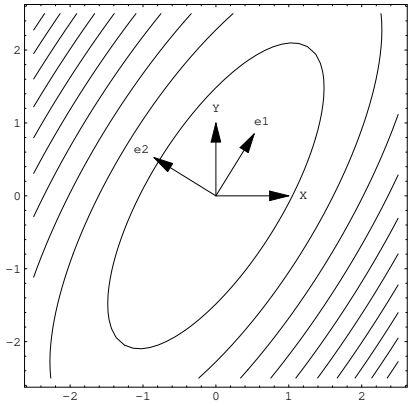


Figure 3.3: Function Contour

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Like the base vectors $\overrightarrow{i}$ and $\overrightarrow{j}$ in the x-y coordinates the eigenvectors are of unit length, *because we made them that way!*

There is another interesting thing about this contour plot. *What is the distance from the origin to the first contour line along the directions of the eigenvectors?* This is a question that we shall investigate now. First we remember our eigenvalue problem statement as, in terms of x and y,

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = \lambda \begin{Bmatrix} x \\ y \end{Bmatrix} \tag{3.78}$$

Now for the problem we just contoured we find,

$$\begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = \lambda \begin{Bmatrix} x \\ y \end{Bmatrix} \tag{3.79}$$

now lets do what the author does on page 91 and 102 when he changes coordinates. In other words if we have a vector $\overrightarrow{r}$ in x-y coordinates we can change coordinate systems to the x' and y' system.

Now from this figure we seek to represent the primed base vectors as a function of the unprimed base vectors. This can be done by use of trigonometry to yield,

$$\begin{aligned} \overrightarrow{i}' &= \cos\theta \overrightarrow{i} + \sin\theta \overrightarrow{j} \\ \overrightarrow{j}' &= -\sin\theta \overrightarrow{i} + \cos\theta \overrightarrow{j} \end{aligned} \tag{3.80}$$

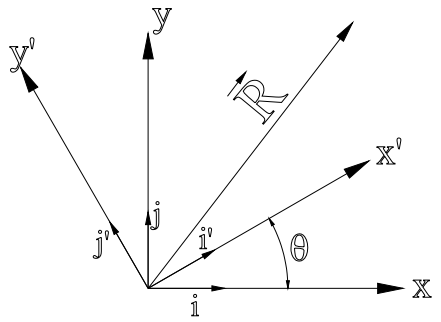


Figure 3.4: Rotation of Coordinates

Now the equation of a any vector $\overrightarrow{R}$ can be written as,

$$\begin{aligned} \overrightarrow{R} &= x \overrightarrow{i} + y \overrightarrow{j} \\ \overrightarrow{R} &= x' \overrightarrow{i}' + y' \overrightarrow{j}' \end{aligned} \tag{3.81}$$

Next we substitute the equation for the primed bases vectors in terms of the unprimed base vectors to get,

$$\begin{aligned} \overrightarrow{R} &= x \overrightarrow{i} + y \overrightarrow{j} \\ \overrightarrow{R} &= x' \left(\cos\theta \overrightarrow{i} + \sin\theta \overrightarrow{j} \right) + y' \left(-\sin\theta \overrightarrow{i} + \cos\theta \overrightarrow{j} \right) \end{aligned} \tag{3.82}$$

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Now from these two equations we can see that a rotation of coordinates gives;

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} x' \\ y' \end{Bmatrix} \tag{3.83}$$

Combining this equation with equation 3.79 we get,

$$\begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} x' \\ y' \end{Bmatrix} = \lambda \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} x' \\ y' \end{Bmatrix} \tag{3.84}$$

Now premultiply by the inverse of the rotation matrix we get,

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} x' \\ y' \end{Bmatrix} = \lambda \begin{Bmatrix} x' \\ y' \end{Bmatrix} \tag{3.85}$$

Now in our case this rotation matrix gave an angle of $\theta = 1.0173$, remember equation 3.57. Now our equation becomes,

$$\begin{bmatrix} 0.5257 & 0.8507 \\ -0.8507 & 0.5257 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 0.5257 & -.8507 \\ 0.8507 & 0.5257 \end{bmatrix} \begin{Bmatrix} x' \\ y' \end{Bmatrix} = \lambda \begin{Bmatrix} x' \\ y' \end{Bmatrix} \tag{3.86}$$

Now from equation 3.59 we get,

$$\begin{bmatrix} 3-\sqrt{5} & 0 \\ 0 & 3+\sqrt{5} \end{bmatrix} \begin{Bmatrix} x' \\ y' \end{Bmatrix} = \begin{bmatrix} 0.763932 & 0 \\ 0 & 5.2360679 \end{bmatrix} \begin{Bmatrix} x' \\ y' \end{Bmatrix} = \lambda \begin{Bmatrix} x' \\ y' \end{Bmatrix} \tag{3.87}$$

Premultiplying this equation the same way we did equation 3.68 we get,

$$\begin{Bmatrix} x' & y' \end{Bmatrix} \begin{bmatrix} 0.763932 & 0 \\ 0 & 5.2360679 \end{bmatrix} \begin{Bmatrix} x' \\ y' \end{Bmatrix} = \lambda \begin{Bmatrix} x' & y' \end{Bmatrix} \begin{Bmatrix} x' \\ y' \end{Bmatrix} = \lambda \tag{3.88}$$

Where we normalized our vector as before, this allowed the right hand side to be equal to λ. This equation is a conic section just like the one before. However the conic section is an actual ellipse with the major axis in the x' direction and the minor axis in the y' direction.

$$f(x', y') = 0.763932(x')^2 + 5.2360679(y')^2 - \lambda \tag{3.89}$$

This function is shown as a contour in figure 3.5. If we rewrite equation 3.89 in the following form

$$1 = \frac{(x')^2}{a^2} + \frac{(y')^2}{b^2} \tag{3.90}$$

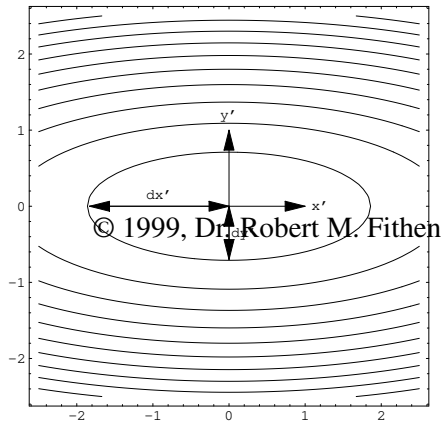


Figure 3.5: Contours

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We can readily see this is the exact equation of an ellipse with major axis of a and minor axis of b. Comparing both equations we see the following,

$$\begin{aligned} a &= \sqrt{\frac{\lambda}{5.2360679}} \\ b &= \sqrt{\frac{\lambda}{0.763932}} \end{aligned} \tag{3.91}$$

In fact if we divide these two equations we get,

$$\frac{b}{a} = \frac{\sqrt{\frac{\lambda}{0.763932}}}{\sqrt{\frac{\lambda}{5.2360679}}} = \sqrt{\frac{5.2360679}{0.763932}} = 2.618034 = \frac{dx'}{dy'} \tag{3.92}$$

This means the ratio of any distance on a contour level, along the eigenvector directions is the square root of the eigenvalues. Or in equation form,

$$\sqrt{\frac{\omega_2^2}{\omega_1^2}} = \frac{dx'}{dy'} \tag{3.93}$$

One thing should be made note of; figure 3.5 is the rotated form of figure 3.4. In other words, x' and y' line up in the direction of the eigenvectors, which can be obtained by rotating figure 3.4 to the right a value of θ.

3.3 Relating Eigen-Problems to Systems of Equations

As we stated in class, n equations and n unknowns can be observed as minimizing a functional surface on n variables. Lets take a quick look at this problem. If we decide to use the same matrix as before, namely;

$$\begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} \tag{3.94}$$

You can verify that the solution to this equation is,

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} 1.5 \\ 2.5 \end{Bmatrix} \tag{3.95}$$

However, what we really want to observe is the function that produced this equation. If we look at the function,

$$f(x, y) = 2x^2 - 2xy + y^2 - x - 2y \tag{3.96}$$

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Now, what value of x and y will minimize this function? In order to determine this value we take the total differential of this function and set it to zero.

$$df(x,y) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0 \tag{3.97}$$

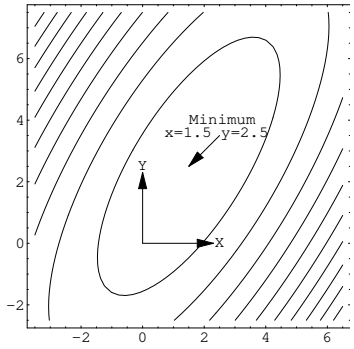


Figure 3.6: Function

Since the values of dx and dy are arbitrary and could themselves be set to zero we get,

$$\begin{aligned} \frac{\partial f}{\partial x} &= 0 \\ \frac{\partial f}{\partial y} &= 0 \end{aligned} \tag{3.98}$$

Taking our function to this point we get,

$$\begin{aligned} \frac{\partial f}{\partial x} &= 4x - 2y - 1 = 0 \\ \frac{\partial f}{\partial y} &= -2x + 2y - 2 = 0 \end{aligned} \tag{3.99}$$

Which indeed gives us back our equation 3.94. Now let's stop now and look at this function and observe its properties. From the contour plot in figure 3.6 we see what appears to be the exact same contour as that shown in figure 3.3. However, the center contour, or the minimum, of the function is shifted. In this case the minimum appears to be located at the solution to the two equations, equation 3.95.

3.3.1 Zero Eigenvalues

Now let's look at some other example equations. First let's look at the following equation.

$$\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} \tag{3.100}$$

Which yields the following function,

$$f(x,y) = 2x^2 - 2xy + \frac{1}{2}y^2 - x - 2y \tag{3.101}$$

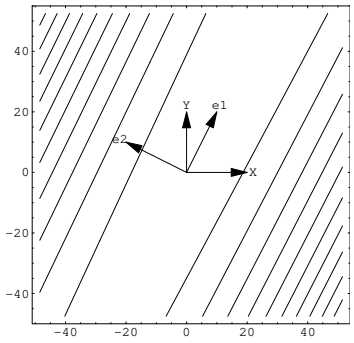


Figure 3.7: Function Contour

Looking at the contour to this function we see something interesting, it is not an ellipse! In fact it appears as though it is a straight line. In fact, a closer look at equation 3.100 reveals that the matrix is singular. Therefore instead of

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two equations and two unknowns, we have one equation and two unknowns. This one equation, from either of the two equations is,

$$4x - 2y = 1 \tag{3.102}$$

Or

$$y = 2x + \frac{1}{2} \tag{3.103}$$

Now an inspection of the contour plot reveals that *All* the contour lines *DO* appear to have a slope of 2. In fact a solution to this equation is the line expressed in equation 3.103! Now that we have established this, what effect does a singular matrix have on its own eigenvalues. Well as we determined the ratio of the eigenvalues is proportional to the square of the distance to any contour along the directions of the eigenvectors! *NOTE: the eigenvectors point along the major and minor axes of the ellipse. Meaning these distances are in fact the min and max!* However, this contour plot has no ellipse. Unless we consider the limit as an ellipse's major axes runs off to infinity. Logically this means that this contour plot implies that one of the eigenvalues is either *zero* or *infinity*. In fact if we find the eigenvalues and eigenvectors of this matrix we get,

$$\begin{aligned} \omega_1^2 = 0 \quad &\longrightarrow \quad \begin{Bmatrix} +0.4472 \\ +0.8894 \end{Bmatrix} \\ \omega_2^2 = 5 \quad &\longrightarrow \quad \begin{Bmatrix} -0.8894 \\ +0.4472 \end{Bmatrix} \end{aligned} \tag{3.104}$$

Looking at the first eigenvector we note that it points in a direction parallel to the contour lines and the second eigenvector point in a direction orthogonal to the contour lines. From all of this we can conclude that, **All singular matrices have at least one zero eigenvalue!**

3.3.2 Negative Eigenvalues

Now let's look at one other example equations.

$$\begin{bmatrix} 4 & -2 \\ -2 & -2 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} \tag{3.105}$$

The function which will yield this matrix is,

$$f(x, y) = 2x^2 - 2xy - y^2 - x - 2y \tag{3.106}$$

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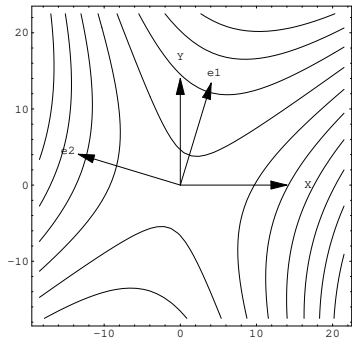


Figure 3.8: Function Contour

From the contour shown in figure 3.8 we see something different. Remember from Calculus what a Hyperbola was, look back at equation 3.73. In fact if we try to find a minimum of this function we can not find a global minimum. We can find a local minimum at the solution point. *The minimum of this function is in fact a saddle point!* A point were if we run along in the direction of one of the eigenvectors we run uphill and if we run along the other eigenvector we run downhill. If we now stop to examine the eigenvalues and eigenvectors we get,

$$\begin{aligned} \omega_1^2 = -2.6056 \quad &\longrightarrow \quad \begin{Bmatrix} +0.2898 \\ +0.9571 \end{Bmatrix} \\ \omega_2^2 = +4.6056 \quad &\longrightarrow \quad \begin{Bmatrix} -0.9571 \\ +0.2898 \end{Bmatrix} \end{aligned} \tag{3.107}$$

The eigenvectors, or at least a scalar time the eigenvectors, are shown in figure 3.8. Along the first eigenvector (the one associate with the negative eigenvalue) the function drops off, or becomes negative, in this direction. In the direction of the second eigenvector the function is increasing.

3.3.3 Matrix and Vector Norms

In this section we investigate the the concepts of matrix and vector norms. Throughout engineering we determine norm of vectors. For any vector $\overrightarrow{r}$ which can be represented by;

$$\overrightarrow{r} = x\,\overrightarrow{i} + y\,\overrightarrow{j} \tag{3.108}$$

we found this vectors magnitude, *or norm* as

$$\|\overrightarrow{r}\| = \sqrt{\overrightarrow{r} \cdot \overrightarrow{r}} = \sqrt{x^2 + y^2} \tag{3.109}$$

However, we can represent this vector length

$$\|\overrightarrow{r}\|^2 = \begin{Bmatrix} x & y \end{Bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} \tag{3.110}$$

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```
clear A=[2 -2;-2 5]
b=[1;3]
syms x y
xy=[x;y]
q=A*xy
xyT=[x y]
fun=xyT*q/2-xyT*b
diff(fun,x)
diff(fun,y)
d=6
[X,Y]=meshgrid([-d:.1:d],[-d:.1:d]);
s=size(X);
for i=1:s(1)
   for j=1:s(1)
      z(i,j)=X(i,j)*(2*X(i,j)-2*Y(i,j))/2+Y(i,j)*(-2*X(i,j)+5*Y(i,j))/2
   end
end
contour(X,Y,z,20)
eig(A)
inv(A)*b
```

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Some properties of vector norms are;

$$\begin{aligned} \|\overrightarrow{x}\| &\geq 0 \\ \|a\overrightarrow{x}\| &= |a|\,\|\overrightarrow{x}\| \text{ for any scalar } a \\ \|\overrightarrow{x}+\overrightarrow{y}\| &\leq \|\overrightarrow{x}\| + \|\overrightarrow{y}\| \end{aligned} \tag{3.111}$$

The last inequality if sometime called the triangle inequality and is use often in mathematical proofs in a wide array of subject matter. For a vector of arbitrary size, say n, the norm is defined as,

$$\overrightarrow{x} = \sqrt{\sum_{i=1}^{n} x_i^2} \tag{3.112}$$

where the x_i's are individual elements of the vector $\overrightarrow{x}$. In an n dimensional space this norm has a property that we have seen from the past. The *dot product* of two vectors is,

$$\overrightarrow{r} \cdot \overrightarrow{s} = (\overrightarrow{r}, \overrightarrow{s}) = \|\overrightarrow{r}\|\,\|\overrightarrow{s}\| \cos\theta \tag{3.113}$$

Where θ is the angle between the two vectors. The notation $(\overrightarrow{r}, \overrightarrow{s})$ is a representation use occasionally in other areas of mathematics. From this dot product equation we recognize that two vectors are orthogonal if

$$(\overrightarrow{r}, \overrightarrow{s}) = 0 \tag{3.114}$$

This means that the angle between these two vectors is 90^o. If we now recall for our example starting at equation 3.8 we had eigenvectors of;

$$\{a_1\} = \begin{Bmatrix} 0.525732 \\ 0.85065 \end{Bmatrix} \tag{3.115}$$

$$\{a_2\} = \begin{Bmatrix} -0.85065 \\ 0.525731 \end{Bmatrix} \tag{3.116}$$

If we attempt to dot the two eigenvectors together we get zero. In fact, for any n dimensional equation we get n eigenvalues and n eigenvectors. All eigenvectors are orthogonal to each other. Why? You have seen the answer. We have already seen that any function from the family of *conic* sections will yield a matrix, if we are looking to minimize that function. We noticed that for a negative, zero and positive eigenvalues the eigenvectors are orthogonal. This makes sense if we consider that an ellipse is a member of the conic section, and the major and minor axis of an ellipse is always orthogonal to each other. In fact for all conic sections,

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even in higher dimensions, this condition hold true. Mathematically this condition can be expressed as,

$$(\overline{a}_i, \overline{a}_j) = 0 \quad \forall \quad i \neq j \tag{3.117}$$

The symbol $\forall$ can be replaced by the words *for all*. Now if we have a vector in two dimensions we can represent this vector as in equation 3.81, repeated here for you convenience.

$$\begin{aligned} \overrightarrow{R} &= x\overrightarrow{i} + y\overrightarrow{j} \\ \overrightarrow{R} &= x'\overrightarrow{i}' + y'\overrightarrow{j}' \end{aligned} \tag{3.118}$$

The $\overrightarrow{i}'$ and $'\overrightarrow{j}'$ we actually the eigenvectors. We may not have known it at that time but you should now know this to be the case. We stop now to consider a definition.

DEFINITION 3.3.1 (*Spanning a vector space*) *Let $\mathcal{S} = \{\overline{a}_1, \overline{a}_2, \ldots, \overline{a}_k\}$ be a set of vectors in space $\mathcal{V}$. The set of vectors $\mathcal{S}$ is said to* span *the vector space $\mathcal{V}$, if every vector that exist in $\mathcal{V}$ can be written as a linear combination of the vectors existing in $\mathcal{S}$*

In order to clarify this definition we look at a three dimensional space. That three dimensional space is actually space $\mathcal{V}$ in the definition. First, in order to span this three dimensional space we need at least three vectors in $\mathcal{S}$. For convenience lets take the three vectors $\overrightarrow{i}$, $\overrightarrow{j}$ and $\overrightarrow{k}$. This is a no-brainer, these three vectors definitely span the space. In fact this is how we represent vectors in three dimensional space,

$$\overrightarrow{R} = x\overrightarrow{i} + y\overrightarrow{j} + z\overrightarrow{k} \tag{3.119}$$

We could represent these vectors and equation as,

$$\begin{Bmatrix} R_x \\ R_y \\ R_z \end{Bmatrix} = x \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} + y \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} + z \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \tag{3.120}$$

Where the vector to the left of the equal sign contains the components of the vector we are interested in. In equation form this could be thought of as,

$$\overrightarrow{R} = \begin{Bmatrix} i & j & k \end{Bmatrix} \begin{Bmatrix} R_x \\ R_y \\ R_z \end{Bmatrix} \tag{3.121}$$

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Now what about another set of vectors $\mathcal{S}$? Lets take an example, will the following set of vectors *span* three dimensional space?

$$\mathcal{S} = \left\{ \begin{Bmatrix} 1 \\ 2 \\ 0 \end{Bmatrix}, \begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix}, \begin{Bmatrix} 3 \\ 0 \\ 0 \end{Bmatrix} \right\} \tag{3.122}$$

We can test this out by picking any arbitrary vector in three dimensional space and trying to represent this vector using these three vectors as reference (or *base*) vectors. Do so we get,

$$\begin{Bmatrix} R_x \\ R_y \\ R_z \end{Bmatrix} = a \begin{Bmatrix} 1 \\ 2 \\ 0 \end{Bmatrix} + b \begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix} + c \begin{Bmatrix} 3 \\ 0 \\ 0 \end{Bmatrix} \tag{3.123}$$

This leads to three equations and three unknowns as,

$$\begin{Bmatrix} R_x \\ R_y \\ R_z \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \end{Bmatrix} \tag{3.124}$$

The solution to this system of equations is

$$\begin{aligned} a &= \frac{R_y}{2} - \frac{R_z}{2} \\ b &= R_x \\ c &= \frac{R_x}{3} - \frac{R_y}{6} + \frac{R_z}{6} \end{aligned} \tag{3.125}$$

Since a solution is possible the vectors do in fact span three dimensional space. The inevitable question would be, what three vectors do not span three dimensional space? If we look at the following three vectors,

$$\mathcal{S} = \left\{ \begin{Bmatrix} 1 \\ 2 \\ 0 \end{Bmatrix}, \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}, \begin{Bmatrix} 3 \\ 0 \\ 0 \end{Bmatrix} \right\} \tag{3.126}$$

To which the matrix equation will become,

$$\begin{Bmatrix} R_x \\ R_y \\ R_z \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \end{Bmatrix} \tag{3.127}$$

There is no solution possible for a, b and c here. The determine of the matrix is zero proving that the matrix is singular and there is no solution. Hence, this set of vectors do not span three dimensional space.

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DEFINITION 3.3.2 (*Linearly Independent Vectors*) *Let $\mathcal{S} = \{\overline{a}_1, \overline{a}_2, \ldots, \overline{a}_k\}$ be a set of vectors. The set of vectors $\mathcal{S}$ is linearly independent vectors if any vector from this set **can not** be expressed as linear combination of all other vectors in this set.*

What exactly does it mean to be linearly independent of another group of vectors. Lets take an example, are the set of vectors,

$$\left\{ \begin{Bmatrix} 1 \\ 2 \\ 0 \end{Bmatrix}, \begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix}, \begin{Bmatrix} 3 \\ 0 \\ 0 \end{Bmatrix} \right\} \tag{3.128}$$

linearly independent? We can determine this by trying to express any one vector as a linear combination of the others.

$$\tilde{c}_1 \begin{Bmatrix} 1 \\ 2 \\ 0 \end{Bmatrix} = c_2 \begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix} + c_3 \begin{Bmatrix} 3 \\ 0 \\ 0 \end{Bmatrix} \tag{3.129}$$

If we can find a set of constants c_1, c_2 and c_3 which are all nonzero we can conclude that we have a linearly dependent set. This means that one vector *is* a linear combination of the other vectors. If we find the only solution for c_1, c_2 and c_3 is one where all constants are zero, we have a linearly independent set of vectors. Writing this equation, with $c_1 = \tilde{c}_1$ we get,

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{3.130}$$

Trying to find a solution to this system we first take the determinate of the system matrix to get,

$$\begin{vmatrix} 1 & 0 & 3 \\ 2 & 1 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 6 \tag{3.131}$$

Therefore Cramer's rule would tell us that the solution will be

$$\begin{aligned} c_1 &= 0 \\ c_2 &= 0 \\ c_3 &= 0 \end{aligned} \tag{3.132}$$

Hence this set of vectors do span three dimensional space.

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DEFINITION 3.3.3 (*Linearly Independent Vectors and Basis Vectors*) *Let $\mathcal{S} = \{\overline{a}_1, \overline{a}_2, \ldots, \overline{a}_k\}$ be a set of vectors that span space $\mathcal{V}$. The set of vectors $\mathcal{S}$ is a set of linearly independent vectors.*

In the last few paragraphs we have seen that the determinate of the matrix plays a crucial role in determining several properties of the base vectors. If the determinate of the matrix is zero, then one vector is linear combination of the other two, and the set of vectors is linearly dependent. If the set of vectors is linearly dependent it can not span the space.

If we now stop and take a closer look at our eigenvectors we see something interesting. We took our eigenvectors and placed them in a matrix exactly like we have done in the last few paragraphs. That method was placing the eigenvectors in columns of the matrix. Our eigenvectors we made into unit length vectors, in other words we normalized the eigenvectors. We have already noted that all eigenvectors are orthogonal to each other.

DEFINITION 3.3.4 (*Eigenvectors from a Symmetric*) *Let $\mathcal{S} = \{\overline{a}_1, \overline{a}_2, \ldots, \overline{a}_n\}$ be a set of normalized eigenvectors. Let space $\mathcal{V}$ be an n dimensional space, the same n that exist as the final subscript in set $\mathcal{S}$. In other words if there are 6 eigenvectors then $\mathcal{V}$ will be a 6 dimensional space. The set of vectors $\mathcal{S}$ is a set of linearly independent vectors which span the space $\mathcal{V}$. In addition the set $\mathcal{S}$ is orthogonal, and its elements are all of unit length. Therefore, the determinate of a matrix constructed of columns of n eigenvectors will always be a value of ± 1. This is all contingent of the matrix being symmetric. UNLuckily, not all vibrating systems give symmetric matrices.*

Lets take an extreme example, consider the matrix,

$$[a] = \begin{bmatrix} 4 & -2 & 0 & 0 & 0 & 0 \\ -2 & 4 & -2 & 0 & 0 & 0 \\ 0 & -2 & 4 & -2 & 0 & 0 \\ 0 & 0 & -2 & 4 & -2 & 0 \\ 0 & 0 & 0 & -2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix} \tag{3.133}$$

From the MatLab command `[v,d]=eig(a)` we get the matrix with eigenvectors

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as columns,

$$[v] = \begin{bmatrix} -0.5000 & 0.5000 & 0.5774 & -0.2887 & 0.2887 & 0 \\ -0.5000 & -0.5000 & 0.0000 & 0.5000 & 0.5000 & 0 \\ 0 & 0.0000 & -0.5774 & -0.5774 & 0.5774 & 0 \\ 0.5000 & 0.5000 & 0.0000 & 0.5000 & 0.5000 & 0 \\ 0.5000 & -0.5000 & 0.5774 & -0.2887 & 0.2887 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.0000 \end{bmatrix} \tag{3.134}$$

If we use the MatLab `det(v)` command the result gives a determinate of -1.

At this point we can conclude that the eigenvectors of a $n \times n$ span n dimensional space. Therefore, *all* vectors in this n dimensional space can be represented as a linear combination of the eigenvectors. Mathematically the equation representing this concept is,

$$\{q\} = \sum_{i=1}^{n} c_i \{e\}_i \tag{3.135}$$

Where c_i is a constant depending on the vector $\{q\}$. Also, $\{e\}_i$ is the eigenvector associated with the i^{th} eigenvalue. In equation form,

$$[A]\{e\}_i = \lambda_i \{e\}_i \tag{3.136}$$

With these concepts we move to looking at the norm of a matrix. The norm of a vector can be thought of as its length. How do we determine the length of a matrix? This does not make sense unless we consider what a matrix is. A matrix could be though of as a mapping tool. If we have a vector r and a matrix A we could consider the matrix to map the vector to another vector,

$$[A]\{r\} = \{b\} \tag{3.137}$$

This mapping can be seen in figure 3.9 If we consider the matrix as a mapping function we can now dream up a matrix norm. For that matter, using this concept we can define several matrix norms. The one we shall concentrate on is that used in MatLab. If we divide the length of the resulting vector b over the original vector r we can call that the norm of the mapping matrix. In equation form this becomes,

$$\text{norm}\,(A) = \left[\frac{|\{b\}|}{|\{r\}|}\right]_{\max} = \left[\frac{|[A]\{r\}|}{|\{r\}|}\right]_{\max} \tag{3.138}$$

Now $\{r\}$ is any arbitrary vector, our task is to look at *all* possible vectors in this n dimensional space in order to maximize the right hand side of this equation. This may at first appear to be a monumental task, however, the fact that all vectors in this

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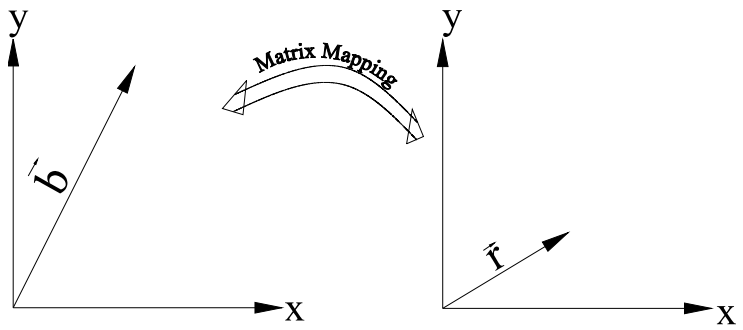


Figure 3.9: Matrix Mapping

n dimensional space can be represented as a linear combination of the eigenvectors of matrix $[A]$ will come in very handy. If we represent the arbitrary vector $\{r\}$ using the expansion of eigenvectors we get,

$$\{r\} = \sum_{i=1}^{n} c_i \{e\}_i \tag{3.139}$$

Where now our c_i's are constants we can adjust in order to get a maximum from,

$$\text{norm}\,(A) = \left[\frac{|\sum_{i=1}^{n} c_i [A] \{e\}_i|}{|\sum_{i=1}^{n} c_i \{e\}_i|} \right]_{\max} \tag{3.140}$$

Now we make use of the general form of the eigen problem as,

$$[A] \{e\}_i = \lambda_i \{e\}_i \tag{3.141}$$

To get the equation,

$$\text{norm}\,(A) = \left[\frac{|\sum_{i=1}^{n} c_i \lambda_i \{e\}_i|}{|\sum_{i=1}^{n} c_i \{e\}_i|} \right]_{\max} = |\lambda_i|_{\max} \tag{3.142}$$

How does this equation equal to the maximum eigenvalue? This can be clarified by use of the triangle inequality. For example if we consider a system in two dimensions we have,

$$\text{norm}\,(A) = \left[\frac{|c_1 \lambda_1 \{e\}_1 + c_2 \lambda_2 \{e\}_2|}{|c_1 \{e\}_1 + c_2 \{e\}_2|} \right]_{\max} \tag{3.143}$$

If we pick $c_1 \neq 0$ and $c_2 = 0$ or $c_1 = 0$ and $c_2 \neq 0$ we get the following,

$$\begin{aligned} \left[\frac{|c_1 \lambda_1 \{e\}_1 + c_2 \lambda_2 \{e\}_2|}{|c_1 \{e\}_1 + c_2 \{e\}_2|} \right]_{c_2=0} &= \lambda_1 \\ \left[\frac{|c_1 \lambda_1 \{e\}_1 + c_2 \lambda_2 \{e\}_2|}{|c_1 \{e\}_1 + c_2 \{e\}_2|} \right]_{c_1=0} &= \lambda_2 \end{aligned} \tag{3.144}$$

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Now lets pick our indices such that $\lambda_1 > \lambda_2$. Must now prove that for any arbitrary choice of c_1 and c_2 our bounds for this norm are between λ_1 and λ_2. In equation form this can be written, *Prove* the following.

$$\lambda_1 \geq \frac{|c_1\lambda_1 \{e\}_1 + c_2\lambda_2 \{e\}_2|}{|c_1 \{e\}_1 + c_2 \{e\}_2|} \geq \lambda_2 \tag{3.145}$$

Now since the norm of the numerator and denominator are both greater than zero we can multiply these terms to get,

$$\lambda_1 |c_1 \{e\}_1 + c_2 \{e\}_2| \geq |c_1\lambda_1 \{e\}_1 + c_2\lambda_2 \{e\}_2| \geq \lambda_2 |c_1 \{e\}_1 + c_2 \{e\}_2| \tag{3.146}$$

We now insert the eigenvalues into the norm of each matrix to get.

$$|c_1\lambda_1 \{e\}_1 + c_2\lambda_1 \{e\}_2| \geq |c_1\lambda_1 \{e\}_1 + c_2\lambda_2 \{e\}_2| \geq |c_1\lambda_2 \{e\}_1 + c_2\lambda_2 \{e\}_2| \tag{3.147}$$

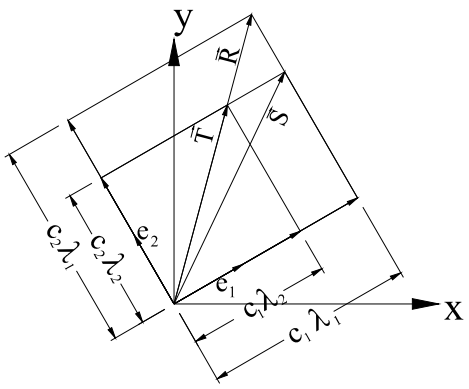


Figure 3.10: Visual Proof

Representing these three vectors on a diagram and defining as,

$$\begin{aligned} \left|\overrightarrow{R}\right| &= |c_1\lambda_1 \{e\}_1 + c_2\lambda_1 \{e\}_2| \\ \left|\overrightarrow{S}\right| &= |c_1\lambda_1 \{e\}_1 + c_2\lambda_2 \{e\}_2| \\ \left|\overrightarrow{T}\right| &= |c_1\lambda_2 \{e\}_1 + c_2\lambda_2 \{e\}_2| \end{aligned} \tag{3.148}$$

From this figure we see first the vectors $\overrightarrow{R}$ and $\overrightarrow{T}$ are colinear. In fact this will always be the case due to the fact that they are just a constant time the same vector. This same vector is in the denominator of the original inequality, equation 3.145. An n dimensional result can be extrapolated from the 2 dimensional result.

Now consider the triangle identity.

$$|c_1\lambda_1 \{e\}_1 + c_2\lambda_2 \{e\}_2| \leq c_1\lambda_1 |\{e\}_1| + c_2\lambda_2 |\{e\}_2| \tag{3.149}$$

This identity can be shown visually in the figure here. We know that the length of the vector $\overrightarrow{r}$ plus the vector $\overrightarrow{s}$ is greater that the length of the sum of the two vectors. The length would be the same is and only if the two vectors were

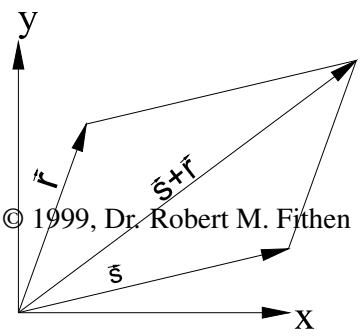


Figure 3.11: Triangle Identity

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colinear. The triangle inequality in this case would be,

$$|\overrightarrow{r} + \overrightarrow{s}| \leq |\overrightarrow{r}| + |\overrightarrow{s}| \tag{3.150}$$

3.4 General Motion of Systems

To this point we only considered systems of equations which have equal masses. However, this is not a general condition of systems. In this section our goal will be to obtain solutions to these dynamical systems of equations through the use of eigenvalue eigenvector analysis. This is commonly called *Modal Analysis*, something that is very crucial for large systems of differential equations. If we return to figure 3.2 we obtain equation 3.63 for the eigen problem, repeated here for convenience.

$$\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} \tag{3.151}$$

Example 3.4.1 (Unequal Mass)
Lets begin by taking an example based on this matrix. With $k_1 = k_2 = 2$ *and* $m_1 = 4$, $m_2 = 2$ *we see a different form of the equation than before.*

$$\begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} \tag{3.152}$$

Returning to the original matrix vector relation we get,

$$\begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix} \begin{Bmatrix} a_{1i} \\ a_{2i} \end{Bmatrix} = \omega_i^2 \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{Bmatrix} a_{1i} \\ a_{2i} \end{Bmatrix} \tag{3.153}$$

Rearranging this equation to one side we get the equation

$$\begin{bmatrix} 4 - 4\omega_i^2 & -2 \\ -2 & 2 - 2\omega_i^2 \end{bmatrix} \begin{Bmatrix} a_{1i} \\ a_{2i} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3.154}$$

As before, in order to obtain a non-trivial solution for the eigenvectors the determinate of the matrix must be forced to zero. This will give the polynomial equation,

$$4 - 16\omega_i^2 + 8\omega_i^4 = 0 \tag{3.155}$$

Yielding the two eigenvalues of,

$$\begin{aligned} \omega_1^2 &= 0.2929 \\ \omega_2^2 &= 1.7071 \end{aligned} \tag{3.156}$$

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Substituting the first eigenvalue back into the equation we get,

$$\begin{bmatrix} 4-4\,(0.2929) & -2 \\ -2 & 2-2\,(0.2929) \end{bmatrix} \begin{Bmatrix} a_{11} \\ a_{21} \end{Bmatrix} = \begin{bmatrix} 2.8284 & -2 \\ -2 & 1.4142 \end{bmatrix} \begin{Bmatrix} a_{11} \\ a_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3.157}$$

which as before gives only one equation for two unknowns. From this equation we can obtain the eigenvector of,

$$\begin{Bmatrix} a_{11} \\ a_{21} \end{Bmatrix} = \begin{Bmatrix} 2 \\ 2.8284 \end{Bmatrix} \tag{3.158}$$

If we normalize this eigenvector to a length of unity we get a vector,

$$\begin{Bmatrix} a_{11} \\ a_{21} \end{Bmatrix} = \begin{Bmatrix} 0.5774 \\ 0.8165 \end{Bmatrix} \tag{3.159}$$

Going through this same exact procedure for the second eigenvalue we get the eigenvector matrix,

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 0.5774 & -0.5774 \\ 0.8165 & 0.8165 \end{bmatrix} \tag{3.160}$$

One interesting property of the eigenvector matrix to note here is its inverse. If we take this matrices inverse we do not get the transpose as before.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \begin{bmatrix} 0.8660 & 0.6124 \\ -0.8660 & 0.6124 \end{bmatrix} \tag{3.161}$$

Why not? We normalized our eigenvectors! Also upon futher investigation we notice that our eigenvectors are no longer orthogonal!

This may become apparent if we return to look at the general equation, repeated here. However, we premultiply the matrix system by the inverse of the mass matrix.

$$\begin{bmatrix} \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix} \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} \tag{3.162}$$

Multiplying out the matrices on the left hand side we obtain,

$$\begin{bmatrix} \frac{k_1+k_2}{m_1} & \frac{-k_2}{m_1} \\ \frac{-k_2}{m_2} & \frac{k_2}{m_2} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} \tag{3.163}$$

Upon close examination of the left hand side of this system we notice that the matrix is **NOT** symmetric unless $m_1 = m_2$. This brings up a new subject because

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all the matrices we investigated to this point have actually been symmetric. If we now use the following representations for convenience,

$$\begin{bmatrix} \frac{k_1+k_2}{m_1} & \frac{-k_2}{m_1} \\ \frac{-k_2}{m_2} & \frac{k_2}{m_2} \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \tag{3.164}$$

And

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \tag{3.165}$$

We can premultiply this equation by the inverse of the eigenvector matrix to obtain,

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} \tag{3.166}$$

If we now post multiply this equation by the inverse of the eigenvector matrix we get,

$$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} = \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \tag{3.167}$$

The matrix $[b]$ is called the *left eigenvector matrix*. In addition the $[a]$ is called the *right eigenvector matrix*. These two matrices are in fact the transpose of each other, if normalized, if and only if the $[K]$ matrix is symmetric. To see this lets take the transpose of this equation, remembering following identity,

$$[[T]\,[R]]^T = [R]^T\,[T]^T \tag{3.168}$$

We can see that our system becomes,

$$\begin{bmatrix} K_{11} & K_{21} \\ K_{12} & K_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix} \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} \tag{3.169}$$

Interestingly enough this equation is exactly the equation given in equation 3.65 if and only if the masses are equal. If the masses are equal then $K_{12} = K_{21}$ hence the matrix is symmetric and the left eigenvectors are the same as the right eigenvectors.

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3.5 Mass Based Orthogonality

We observed from the last section that our eigenvectors are no longer orthogonal to each other. Why? If we look at the general two dimensional case we can determine why. Returning to the general eigenvalue problem we see the following statement,

$$\begin{bmatrix} k_{11} - m_1\omega_i^2 & k_{12} \\ k_{12} & k_{22} - m_2\omega_i^2 \end{bmatrix} \begin{Bmatrix} a_{1i} \\ a_{2i} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3.170}$$

Remember the goal is the attainment of a nontrivial solution to the constants a_{1i} and a_{2i} by forcing this matrix to be singular. Forcing the matrix's determinate to zero will accomplish this,

$$\begin{vmatrix} k_{11} - m_1\omega_i^2 & k_{12} \\ k_{12} & k_{22} - m_2\omega_i^2 \end{vmatrix} = 0 \tag{3.171}$$

The characteristic equation from this determinate becomes,

$$\left(k_{11} - m_1\omega_i^2\right)\left(k_{22} - m_2\omega_i^2\right) - k_{12}^2 = 0 \tag{3.172}$$

Rearranging,

$$m_1 m_2 \omega_i^4 - \left(k_{11}m_2 + m_1 k_{22}\right)\omega_i^2 + k_{11}k_{22} - k_{12}^2 = 0 \tag{3.173}$$

Using the quadratic formula to find the roots of this equation we obtain,

$$\omega_i^2 = \frac{k_{11}m_2 + m_1 k_{22}}{2m_1 m_2} \pm \frac{\sqrt{\left(k_{11}m_2 + m_1 k_{22}\right)^2 - 4m_1 m_2\left(k_{11}k_{22} - k_{12}^2\right)}}{2m_1 m_2} \tag{3.174}$$

Now since the entire point of this exercise is to render the matrix singular by choice of an appropriate value for ω_i, there exist only one equation and two unknowns from equation 3.185. Taking the first entry in the first eigenvector as unity, $a_{11} = 1$, and the second value in the second eigenvalue as unity, $a_{22} = 1$ we obtain the general equation for the second entry,

$$\begin{aligned} \left(k_{11} - m_1\omega_i^2\right) a_{1i} &= -k_{12}a_{2i} \\ \left(k_{22} - m_2\omega_j^2\right) a_{2j} &= -k_{12}a_{1j} \end{aligned} \tag{3.175}$$

Remembering $a_{11} = a_{22} = 1$ we get, setting $i = 1$ and $j = 2$,

$$\begin{aligned} a_{21} &= \frac{\left(m_1\omega_1^2 - k_{11}\right)}{k_{12}} \\ a_{2i} &= \frac{\left(m_2\omega_2^2 - k_{22}\right)}{k_{12}} \end{aligned} \tag{3.176}$$

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Hence our eigenvectors become,

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & \frac{\left(m_2\omega_2^2 - k_{22}\right)}{k_{12}} \\ \frac{\left(m_1\omega_1^2 - k_{11}\right)}{k_{12}} & 1 \end{bmatrix} \tag{3.177}$$

Now we investigate an interesting property of eigenvectors, their orthogonality related to the mass matrix. If we observe the following equation,

$$\begin{aligned} &\begin{Bmatrix} a_{12} & a_{22} \end{Bmatrix} \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} a_{11} \\ a_{21} \end{Bmatrix} \\ &\qquad = \begin{Bmatrix} \frac{(m_2\omega_2 - k_{22})}{k_{12}} & 1 \end{Bmatrix} \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} 1 \\ \frac{(m_1\omega_1 - k_{11})}{k_{12}} \end{Bmatrix} \\ &\qquad = \begin{Bmatrix} \frac{(m_2\omega_2 - k_{22})}{k_{12}} & 1 \end{Bmatrix} \begin{Bmatrix} m_1 \\ \frac{m_2(m_1\omega_1 - k_{11})}{k_{12}} \end{Bmatrix} \\ &\qquad = \frac{m_1\left(m_2\omega_2 - k_{22}\right)}{k_{12}} + \frac{m_2\left(m_1\omega_1 - k_{11}\right)}{k_{12}} \end{aligned} \tag{3.178}$$

Noticing from equation 3.179 the two frequencies can be written as,

$$\begin{aligned} \omega_1^2 &= \frac{k_{11}m_2 + m_1k_{22}}{2m_1m_2} - \mathcal{A} \\ \omega_2^2 &= \frac{k_{11}m_2 + m_1k_{22}}{2m_1m_2} + \mathcal{A} \end{aligned} \tag{3.179}$$

Where the $\mathcal{A}$ is the radical term and is the same in both equations. Inserting each of these frequencies into the equation we get,

$$\begin{aligned} &\begin{Bmatrix} a_{12} & a_{22} \end{Bmatrix} \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} a_{11} \\ a_{21} \end{Bmatrix} \\ &\qquad = \frac{m_1\left(m_2\left(\frac{k_{11}m_2 + m_1k_{22}}{2m_1m_2} + \mathcal{A}\right) - k_{22}\right)}{k_{12}} \\ &\qquad + \frac{m_2\left(m_1\left(\frac{k_{11}m_2 + m_1k_{22}}{2m_1m_2} - \mathcal{A}\right) - k_{11}\right)}{k_{12}} \\ &\qquad = \frac{k_{11}m_2 + m_1k_{22} - m_1k_{22} - m_2k_{11}}{k_{12}} \\ &\qquad = 0 \end{aligned} \tag{3.180}$$

This proves, that each eigenvector is orthogonal with respect to the mass matrix. In addition the eigenvectors are orthogonal with respect to the stiffness matrices.

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Consider the equation relating to the first eigenvalue,

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} a_{11} \\ a_{21} \end{Bmatrix} = \omega_1^2 \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} a_{11} \\ a_{21} \end{Bmatrix} \tag{3.181}$$

Now, if we premulitiply this equation by the transpose of the second eigenvector we obtain,

$$\begin{Bmatrix} a_{12} & a_{22} \end{Bmatrix} \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} a_{11} \\ a_{21} \end{Bmatrix} = \omega_1^2 \begin{Bmatrix} a_{12} & a_{22} \end{Bmatrix} \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} a_{11} \\ a_{21} \end{Bmatrix} = 0 \tag{3.182}$$

The right hand side of this equation is zero, as proved before. Therefore, the left hand side is also zero and the eigenvectors are orthogonal with respect to the stiffness matrix.

Now we investigate the magnitude of the eigenvectors. With a unit mass matrix we forced our eigenvectors to have unit length. However, with unequal masses another measure of eigenvector length becomes convenient. Since the eigenvectors are orthogonal with respect to the mass matrix we obtain,

$$\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \tag{3.183}$$

If we divide the first eigenvector by $\sqrt{d_1}$ and the second eigenvector by $\sqrt{d_1}$ our new normalized eigenvectors we yield,

$$\begin{bmatrix} \frac{a_{11}}{\sqrt{d_1}} & \frac{a_{21}}{\sqrt{d_1}} \\ \frac{a_{12}}{\sqrt{d_2}} & \frac{a_{22}}{\sqrt{d_2}} \end{bmatrix} \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \frac{a_{11}}{\sqrt{d_1}} & \frac{a_{12}}{\sqrt{d_2}} \\ \frac{a_{21}}{\sqrt{d_1}} & \frac{a_{22}}{\sqrt{d_2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{3.184}$$

3.6 Modal Analysis

This section if the focal point of this chapter. Our main interest to the point has been to build an understanding of the significance of the eigen problem. In this section we obtain a complete understanding of the power of this type of analysis. We shall continually reference previous sections in every step that takes place. We begin by returning to our coupled differential equation. However, This time we place random forces on each mass as shown in figure **??**. Using Newton's second law for each mass we obtain the following equation.

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} = \begin{Bmatrix} F_1(t) \\ F_2(t) \end{Bmatrix} \tag{3.185}$$

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Where, as before,

$$\begin{aligned} k_{11} &= k_1 + k_2 \\ k_{12} &= -k_2 \\ k_{22} &= k_2 \end{aligned} \tag{3.186}$$

If we now consider a transformation equation which transforms from the x_i coordinate system to a new coordinate system η_i thorough the following transformation,

$$\begin{aligned} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{Bmatrix} \eta_1 \\ \eta_2 \end{Bmatrix} \\ \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{Bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{Bmatrix} \end{aligned} \tag{3.187}$$

Where the $[a]$ matrix is the matrix of eigenvectors. Using this transformation in the differential equation we obtain the following differential equation,

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{Bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{Bmatrix} \eta_1 \\ \eta_2 \end{Bmatrix} = \begin{Bmatrix} F_1(t) \\ F_2(t) \end{Bmatrix} \tag{3.188}$$

If we now premultiply this equation by the transpose of the eigenvector matrix we obtain,

$$\begin{aligned} \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{Bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{Bmatrix} + \\ \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{Bmatrix} \eta_1 \\ \eta_2 \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} \begin{Bmatrix} F_1(t) \\ F_2(t) \end{Bmatrix} \end{aligned} \tag{3.189}$$

We now make use of the mass orthogonality property of the eigenvectors. If we normalize the eigenvectors based on the mass matrix our equation becomes,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{Bmatrix} + \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} \begin{Bmatrix} \eta_1 \\ \eta_2 \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} \begin{Bmatrix} F_1(t) \\ F_2(t) \end{Bmatrix} \tag{3.190}$$

Where the mass based orthogonality and mass normalized eigenvector matrix gives the relation,

$$\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{3.191}$$

However, remember the relation,

$$\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{3.192}$$

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Which yields the following relationship,

$$\begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{3.193}$$

Example 3.6.1
Let's consider an example with two degrees of freedom. Consider the stiffness matrix as,

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix} \tag{3.194}$$

And the mass matrix becomes,

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 8 \end{bmatrix} \tag{3.195}$$

In Matlab these entries are accomplished via, Let's now consider the right hand

```
k=[4 -2; -2 4]
m=[4 0;0 8]
```

side forcing function as

$$\begin{Bmatrix} F_1(t) \\ F_2(t) \end{Bmatrix} = \begin{Bmatrix} \sin(\omega_f t) \\ 0 \end{Bmatrix} \tag{3.196}$$

If we now search for the eigenvalues and eigenvectors for this problem we get,

```
EDU>> [v,d]=eig(k,m)

v =

    0.9391    0.5907
   -0.3437    0.8069


d =

    1.1830         0
         0    0.3170
```

We now mass orthogonality property of the eigenvectors,

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```
EDU>> temp=v'*m*v

temp =

    4.4726    0.0000
    0.0000    6.6043
```

Which lets us know that although the eigenvectors are orthogonal to the mass they are not orthonormal. We now seek to force orthogonality by executing the following statements in Matlab. Here we create a new eigenvectors matrix which is orthonormal with respect to the mass matrix.

```
EDU>> vv(:,1)=v(:,1)/sqrt(temp(1,1))

vv =

    0.4440
   -0.1625

EDU>> vv(:,2)=v(:,2)/sqrt(temp(2,2))

vv =

    0.4440    0.2299
   -0.1625    0.3140

EDU>> vv'*m*vv

ans =

    1.0000    0.0000
    0.0000    1.0000
```

As we see the new eigenvector matrix is orthonormal with respect to the mass matrix. Since the eigenvectors are of arbitrary length the new eigenvector matrix is the one we shall use. Using this matrix we can show this matrix is also orthogonal with respect to the stiffness matrix as well.

```
EDU>> vv'*k*vv

ans =
```

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```
    1.1830          0
    0.0000     0.3170

EDU>> d

d =

    1.1830          0
         0     0.3170
```

Notice the diagonals of this matrix become the eigenvector of the original problem. With this information our equations become,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{Bmatrix} + \begin{bmatrix} 1.1830 & 0 \\ 0 & 0.3170 \end{bmatrix} \begin{Bmatrix} \eta_1 \\ \eta_2 \end{Bmatrix} = \begin{bmatrix} 0.4440 & -0.1625 \\ 0.2299 & 0.3140 \end{bmatrix} \begin{Bmatrix} \sin(\omega_f t) \\ 0 \end{Bmatrix} \tag{3.197}$$

Now since the left hand sides of the equations are decoupled we can now investigate the general solution of each of these equations.

$$\begin{aligned} \ddot{\eta}_1 + 1.1830\eta_1 &= 0.4440 \sin(\omega_f t) \\ \ddot{\eta}_2 + 0.3170\eta_2 &= 0.2299 \sin(\omega_f t) \end{aligned} \tag{3.198}$$

Now we consider the general case,

$$\ddot{\eta}_i(t) + \omega_i^2 \eta_i(t) = f_i \sin \omega_f t \tag{3.199}$$

To which the solution we obtained in chapter 2 as,

$$\eta_i(t) = \left(\frac{\dot{\eta}_{i0}}{\omega_i} - \frac{f_i \omega_f}{\omega_i \left(\omega_i^2 - \omega_f^2 \right)} \right) \sin \omega_i t + \eta_{i0} \cos \omega_i t + \frac{f_i}{\omega_i^2 - \omega_f^2} \sin \omega_f t \tag{3.200}$$

Where, $\dot{\eta}_{i0}$ and η_{i0} are the initial conditions. Let's assume the following data applies to our problem,

$$\begin{aligned} x_1(t=0) &= 1 \\ x_2(t=0) &= .5 \\ \dot{x}_1(t=0) &= 0 \\ \dot{x}_2(t=0) &= 2 \\ \omega_f &= 1 \ \textit{Rad / Sec} \end{aligned} \tag{3.201}$$

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Now that we have the initial conditions of of our system we seek the initial conditions in Modal Coordinates. *To determine this value we use our eigenvector matrix as in equation 3.187.*

$$\begin{aligned}\begin{Bmatrix}\eta_1\\ \eta_2\end{Bmatrix}_{t=0} &= \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}^{-1}\begin{Bmatrix}x_1\\ x_2\end{Bmatrix}_{t=0}\\ \begin{Bmatrix}\dot{\eta}_1\\ \dot{\eta}_2\end{Bmatrix}_{t=0} &= \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}^{-1}\begin{Bmatrix}\dot{x}_1\\ \dot{x}_2\end{Bmatrix}_{t=0}\end{aligned} \tag{3.202}$$

Determining this in Matlab can be done as follows,

```
EDU>> x0=[1;.5]

x0 =

    1.0000
    0.5000

EDU>> v0=[0;2]

v0 =

     0
     2

EDU>> vvinv=inv(vv)

vvinv =

    1.7761   -1.3002
    0.9194    2.5119

EDU>> eta0=vvinv*x0

eta0 =

    1.1260
    2.1753

EDU>> etad0=vvinv*v0
```

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```
etad0 =

   -2.6005
    5.0237
```

Now we seek the solution in modal coordinates. In order to accomplish this we seek the constants in equation 3.203. If we rewrite this equation so its constants can be easily observed

$$\eta_i(t) = A1 \sin \omega_i t + A2 \cos \omega_i t + A3 \sin \omega_f t \tag{3.203}$$

these constants will be sought in Matlab.

In Matlab these are obtained by,

```
EDU>> omegaf=1.

omegaf =

     1

EDU>> rhs=[1;0]

rhs =

     1
     0

EDU>> f=vv'*rhs

f =

    0.4440
    0.2299

EDU>> omega=[sqrt(d(1,1));sqrt(d(2,2))]

omega =

    1.0877
    0.5630
```

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```
EDU>> A1=[
   etad0(1)/omega(1)-f(1)*omegaf/omega(1)/(omega(1)^2-omegaf^2);
   etad0(2)/omega(2)-f(2)*omegaf/omega(2)/(omega(2)^2-omegaf^2)]

A1 =

   -4.6216
    8.5002

EDU>> A2=eta0

A2 =

    1.1260
    2.1753

EDU>> A3=[ f(1)/(omega(1)^2-omegaf^2);f(2)/(omega(2)^2-omegaf^2)]

A3 =

    2.4263
    0.2380

EDU>> t=[0:.1:100];

EDU>> eta=[ A1(1)*sin(omega(1)*t)+
      A2(1)*cos(omega(1)*t)+A3(1)*sin(omegaf*t);
      A1(2)*sin(omega(2)*t)+
      A2(2)*cos(omega(2)*t)+A3(2)*sin(omegaf*t)];

EDU>> plot(t,eta)
```

The last statement gives a plot of the resulting Modal displacements, Shown here The only task remaining is the transformation back into physical coordinates, from modal coordinates. This is accomplished in Matlab by the following,

```
EDU>> t=[0:.01:20];

EDU>> eta=[ A1(1)*sin(omega(1)*t)+
```

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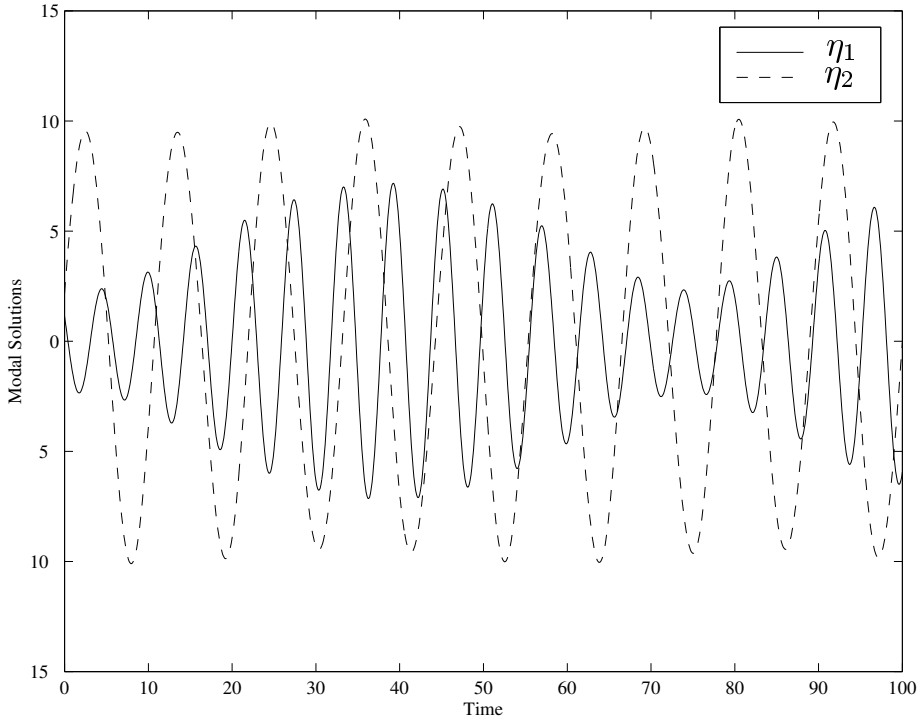


Figure 3.12: Modal Displacements

```
A2(1)*cos(omega(1)*t)+A3(1)*sin(omegaf*t);A1(2)*sin(omega(2)*t)+
A2(2)*cos(omega(2)*t)+A3(2)*sin(omegaf*t)];

EDU>> solution=vv*eta;

EDU>> plot(t,solution)
```

Which produces the physical displacements. Notice our initial conditions are matched. In addition, the beat which is visible in the modal coordinated is not visible in physical coordinates, even though it does occur.

This Completes The Example

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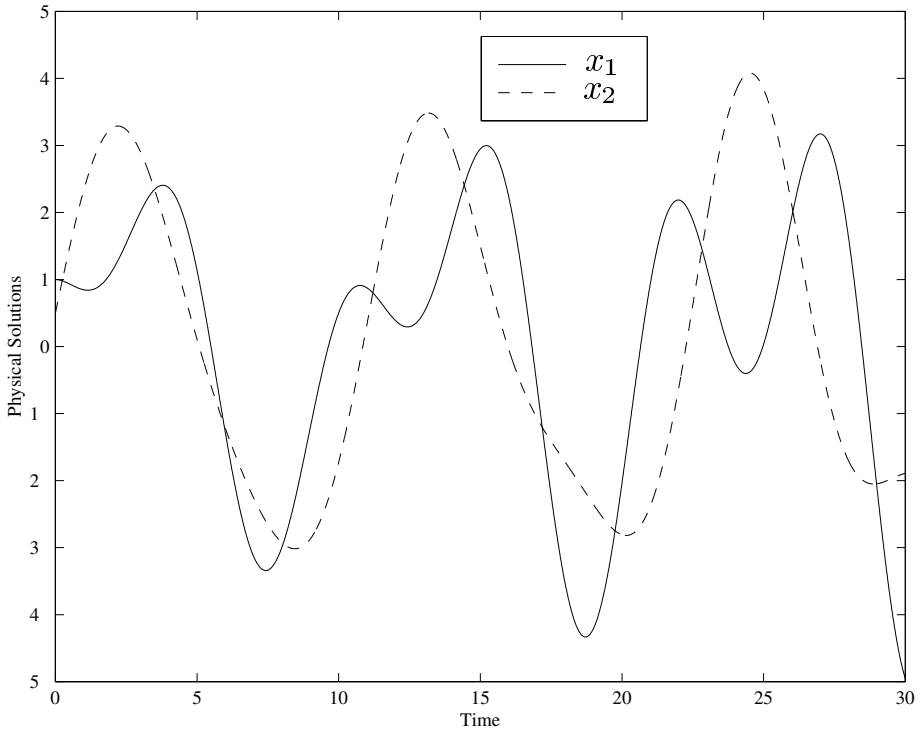


Figure 3.13: Physical Displacements

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Chapter 4

General Forced Vibration

4.1 Periodic Function

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega_T t + b_n \sin n\omega_T t) \tag{4.1}$$

Where

$$\omega_T = \frac{2\pi}{T} \tag{4.2}$$

Where using orthogonality statement,

$$\int_{-T/2}^{T/2} \sin\left(\frac{2n\pi t}{T}\right) \sin\left(\frac{2m\pi t}{T}\right) = \begin{cases} 0 & n \neq m \\ \frac{T}{2} & n = m \end{cases} \tag{4.3}$$

Also Note,

$$\int_{-T/2}^{T/2} \sin\left(\frac{2n\pi t}{T}\right) \cos\left(\frac{2m\pi t}{T}\right) = 0 \tag{4.4}$$

Hence we get

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos n\omega_T t \; dt \quad \forall \; n = 0, 1, 2, 3, \ldots \tag{4.5}$$

$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin n\omega_T t \; dt \quad \forall \; n = 1, 2, 3, \ldots \tag{4.6}$$

Because the differential equation is linear a solution can be obtained via the superpositioning principle,

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega_T t + b_n \sin n\omega_T t) \tag{4.7}$$

Now the particular solution to this equation is the sum of all the particular solutions,

$$\begin{aligned} x(t) &= \sum_{n=0}^{\infty} x_p(t) \\ &= \sum_{n=0}^{\infty} A_n \left(a_n \cos\left(n\omega_T t - \phi_n\right) + b_n \sin\left(n\omega_T t - \phi_n\right)\right) \end{aligned} \tag{4.8}$$

Where,

$$A_n = \frac{1}{\sqrt{\left(\omega_n^2 - n^2\omega_T^2\right)^2 + \left(2\zeta n\omega_T\omega_n\right)^2}} \tag{4.9}$$

$$\phi_n = \tan^{-1}\left(\frac{2\zeta n\omega_T\omega_n}{\omega_n^2 - n^2\omega_T^2}\right) \tag{4.10}$$

Lets assume we have the following right hand side,

$$f(t) = \begin{cases} +1 & \text{if} \quad 0 < t < T/2 \\ -1 & \text{if} \quad -T/2 < t < 0 \end{cases} \tag{4.11}$$

since the function is odd the cosine components of this function are all zero, hence we only have b_n components.

$$\begin{aligned} b_n &= \frac{2}{T}\left(\int_{-T/2}^{0} (-1)\sin n\omega_T t\, dt \int_{0}^{T/2} (+1)\sin n\omega_T t\, dt\right) \\ &= \frac{4}{T}\int_{0}^{T/2} (+1)\sin n\omega_T t\, dt \\ &= \frac{4}{Tn\omega_T}\left(1 - \cos\left(n\omega_T T/2\right)\right) \end{aligned} \tag{4.12}$$

Putting in the $\omega_T = 2\pi/T$ we get

$$b_n = \frac{2}{n\pi}\left(1 - \cos\left(n\pi\right)\right) = \frac{2}{n\pi}\left(1 - (-1)^n\right) \tag{4.13}$$

Hence the square wave loading can be represented as,

$$f(t) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[1 - (-1)^n\right]\sin\left(n\omega_T t\right) \tag{4.14}$$

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