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Spectral Element Methods in Fluids

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NSFIT3D

Abstract

NSFIT2D (NSFIT3D) software package simulates both two and three dimensional incompressible fluid flows. In addition, the software will handle convection and conduction heat transfer as well as turbulent flows through a k-ϵ model (not covered in this talk). In this presentation we shall discuss its formulation and implementation, as well as, several simulation results. NSFIT2D (NSFIT3D) contains two separate, yet distinct, formulations selectable in the main input file. The purpose of these formulation is dealing with the incompressibility constraint. First is the approximate factorization method of Dukowicz and Dvinsky. The second formulation is based in a penalized formulation of the Navier-Stokes equations. In this talk we will cover the penalized formulation of Navier-Stokes equations. We shall first consider the finite element based Stokes formulation followed by the inclusion of convective terms through a Runge-Kutta scheme.

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1. Stokes Equations

For simplicity the formulation show here is two dimensional. Stokes equation may be presented in as,

$$\rho \frac{\partial u}{\partial t} = f_x - \frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \tag{1}$$

$$\rho \frac{\partial v}{\partial t} = f_y - \frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \tag{2}$$

These momentum equations are subject to the incompressibility constraint, continuity equation,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3}$$

Subject to the natural boundary conditions:

$$\mathcal{T}_x = \left[\mu \frac{\partial u}{\partial x} - p \right] n_x + \mu \frac{\partial u}{\partial y} n_y \tag{4}$$

$$\mathcal{T}_y = \mu \frac{\partial v}{\partial x} n_x + \left[\mu \frac{\partial v}{\partial y} - p \right] n_y \tag{5}$$

and to the essential boundary conditions,

$$u\left(x, y, t\right) = u_s \tag{6}$$

$$v\left(x, y, t\right) = v_s \tag{7}$$

as well as the initial conditions,

$$u\left(x, y, 0\right) = u_{ic} \tag{8}$$

$$v\left(x, y, 0\right) = v_{ic} \tag{9}$$

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1.1. Weighted Residual Formulation

These equations may be cast into a weighted residual statement as,

$$0 = \int_x \int_y \delta u \left[\rho \frac{\partial u}{\partial t} - f_x + \frac{\partial p}{\partial x} - \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \right] dxdy \tag{10}$$

$$0 = \int_x \int_y \delta v \left[\rho \frac{\partial v}{\partial t} - f_y + \frac{\partial p}{\partial y} - \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \right] dxdy \tag{11}$$

$$0 = \int_x \int_y \delta p \left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right] dxdy \tag{12}$$

Integrating by parts yields...

$$0 = \int_x \int_y \left[\delta u \left(\rho \frac{\partial u}{\partial t} - f_x \right) - \frac{\partial \delta u}{\partial x} p + \mu \left(\frac{\partial \delta u}{\partial x} \frac{\partial u}{\partial x} + \frac{\partial \delta u}{\partial y} \frac{\partial u}{\partial y} \right) \right] dxdy - \oint \delta u \mathcal{T}_x dS \tag{13}$$

$$0 = \int_x \int_y \left[\delta v \left(\rho \frac{\partial v}{\partial t} - f_y \right) - \frac{\partial \delta v}{\partial y} p + \mu \left(\frac{\partial \delta v}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial \delta v}{\partial y} \frac{\partial v}{\partial y} \right) \right] dxdy - \oint \delta u \mathcal{T}_y dS \tag{14}$$

$$0 = \int_x \int_y \delta p \left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right] dxdy \tag{15}$$

1.2. Finite Element Formulation

$$u = \sum_{j=1}^{n} \psi_j^e u_j \tag{16}$$

$$v = \sum_{j=1}^{n} \psi_j^e v_j \tag{17}$$

$$p = \sum_{j=1}^{m} \phi_j^e p_j \tag{18}$$

$$\delta u = \psi_j^e \tag{19}$$

$$\delta v = \psi_j^e \tag{20}$$

$$\delta p = \phi_j^e \tag{21}$$

Will give a matrix like.....

$$\begin{bmatrix} [M] & [0] & [0] \\ [0] & [M] & [0] \\ [0] & [0] & [0] \end{bmatrix} \begin{bmatrix} (\dot{u}) \\ (\dot{v}) \\ (\dot{p}) \end{bmatrix} + \begin{bmatrix} [G^{11}] & [G^{12}] & [G^{13}] \\ [G^{21}] & [G^{22}] & [G^{23}] \\ [G^{31}] & [G^{32}] & [G^{33}] \end{bmatrix} \begin{bmatrix} (u) \\ (v) \\ (p) \end{bmatrix} = \begin{bmatrix} (F^u) \\ (F^v) \\ (0) \end{bmatrix} \tag{22}$$

Where

$$M_{ij} = \int_x \int_y \rho \psi_i \psi_j dx dy \tag{23}$$

$$G_{ij}^{11} = G_{ij}^{22} = \int_x \int_y \mu \left(\frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} \right) dx dy \tag{24}$$

$$G_{ij}^{12} = 0 \tag{25}$$

$$G_{ij}^{21} = 0 \tag{26}$$

$$G_{ij}^{13} = -\int_x \int_y \frac{\partial \psi_i}{\partial x} \phi_j dx dy \tag{27}$$

$$G_{ij}^{23} = -\int_x \int_y \frac{\partial \psi_i}{\partial y} \phi_j dx dy \tag{28}$$

$$G_{ij}^{13} = G_{ji}^{31} \tag{29}$$

$$G_{ij}^{23} = G_{ji}^{32} \tag{30}$$

$$G_{ij}^{33} = 0 \tag{31}$$

$$F_i^u = \int_x \int_y f_x \psi_i dxdy + \oint \psi_i \left\{ \left(\mu \frac{\partial u}{\partial x} - p \right) n_x + \mu \frac{\partial u}{\partial y} n_y \right\} dS \tag{32}$$

$$F_i^v = \int_x \int_y f_y \psi_i dxdy + \oint \psi_i \left\{ \mu \frac{\partial v}{\partial x} n_x + \left(\mu \frac{\partial v}{\partial y} - p \right) n_y \right\} dS \tag{33}$$

If we rewrote our matrix system 22 to look like,

$$[\mathcal{M}^e] (\dot{\mathcal{U}}) + [\mathcal{K}^e] (\mathcal{U}) - [\mathcal{C}^e] (\mathcal{P}) = (\mathcal{F}^e) \tag{34}$$

$$[\mathcal{C}^e]^T (\mathcal{U}) + \frac{1}{\epsilon} [M_g^e] (\mathcal{P}) = 0 \tag{35}$$

Where

$$[\mathcal{M}^e] = \begin{bmatrix} [M] & [0] \\ [0] & [M] \end{bmatrix} \tag{36}$$

$$[\mathcal{K}^e] = \begin{bmatrix} \begin{bmatrix} G^{11} \\ G^{21} \end{bmatrix} & \begin{bmatrix} G^{12} \\ G^{22} \end{bmatrix} \end{bmatrix} \tag{37}$$

$$[\mathcal{C}^e] = - \begin{bmatrix} \begin{bmatrix} G^{13} \\ G^{23} \end{bmatrix} \end{bmatrix} \tag{38}$$

$$[M_g^e] \Longrightarrow \int_x \int_y \phi_i \phi_j dxdy \tag{39}$$

and ϵ is a very large arbitrary number, the penalty parameter. Rearrange equation 34 we get an equation devoid of the pressure term.

$$(\mathcal{P}) = -\epsilon [M_g^e]^{-1} [\mathcal{C}^e]^T (\mathcal{U}) \tag{40}$$

or

$$[\mathcal{M}^e](\dot{\mathcal{U}}) + [\mathcal{K}^e](\mathcal{U}) + \epsilon [\mathcal{C}^e] \left[M_g^e\right]^{-1} [\mathcal{C}^e]^T (\mathcal{U}) = (\mathcal{F}^e) \tag{41}$$

From this equation the penalty term may easily seen as

$$\left[\mathcal{K}^{Pen}\right] = [\mathcal{C}^e] \left[M_g^e\right]^{-1} [\mathcal{C}^e]^T \tag{42}$$

We may now rewrite our equation into a more reasonable form as,

$$[\mathcal{M}^e](\dot{\mathcal{U}}) + [\widetilde{\mathcal{K}^e}](\mathcal{U}) = (\mathcal{F}^e) \tag{43}$$

Where,

$$[\widetilde{\mathcal{K}^e}] = [\mathcal{K}^e] + \epsilon [\mathcal{C}^e] \left[M_g^e\right]^{-1} [\mathcal{C}^e]^T \tag{44}$$

Assembling these element matrices we obtain a global system of equations,

$$[\mathcal{M}](\dot{\mathcal{U}}) + [\widetilde{\mathcal{K}}](\mathcal{U}) = (\mathcal{F}) \tag{45}$$

1.3. 2nd Order Time Stepping

We use a Crank-Nicholson method to time step our Stokes equations,

$$\frac{(\mathcal{U})^{n+1} - (\mathcal{U})^n}{\Delta t} = \frac{1}{2}\left\{\dot{\mathcal{U}}\right\}^n + \frac{1}{2}\left\{\dot{\mathcal{U}}\right\}^{n+1} \tag{46}$$

Substituting yields,

$$\left[[\mathcal{M}] + \frac{\Delta t}{2}[\widetilde{\mathcal{K}}]\right]\{\mathcal{U}\}^{n+1} = \left[[\mathcal{M}] - \frac{\Delta t}{2}[\widetilde{\mathcal{K}}]\right]\{\mathcal{U}\}^n + \frac{\Delta t}{2}\left\{\{\mathcal{F}\}^n + \{\mathcal{F}\}^{n+1}\right\} \tag{47}$$

1.4. Higher Order Time Stepping (Adams Method)

$$\frac{(\mathcal{U})^{n+1} - (\mathcal{U})^n}{\Delta t} = \beta_0\left\{\ddot{\mathcal{U}}\right\}^{n+1} + \beta_1\left\{\dot{\mathcal{U}}\right\}^n + \beta_2\left\{\dot{\mathcal{U}}\right\}^{n-1} + \beta_3\left\{\dot{\mathcal{U}}\right\}^{n-2} \tag{48}$$

By inserting this relationship into Taylor's series we obtain the unknown β's.

m	β_0	β_1	β_2	β_3	Truncation
1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	second
2	$\frac{5}{12}$	$\frac{8}{12}$	$-\frac{1}{12}$	0	third
3	$\frac{9}{12}$	$\frac{19}{12}$	$-\frac{5}{24}$	$-\frac{1}{24}$	fourth

Fourth order accurate representation of Stokes equations becomes

$$\begin{aligned}\left[[\mathcal{M}]+\beta_0\Delta t[\widetilde{\mathcal{K}}]\right]\{\mathcal{U}\}^{n+1} &= \left[[\mathcal{M}]-\beta_1\Delta t[\widetilde{\mathcal{K}}]\right]\{\mathcal{U}\}^{n} \\ &-\beta_2\Delta t[\widetilde{\mathcal{K}}]\{\mathcal{U}\}^{n-1}-\beta_3\Delta t[\widetilde{\mathcal{K}}]\{\mathcal{U}\}^{n-2} \\ &+\Delta t\left\{\beta_0\{\mathcal{F}\}^{n+1}+\beta_1\{\mathcal{F}\}^{n}+\beta_2\{\mathcal{F}\}^{n-1}+\beta_3\{\mathcal{F}\}^{n-2}\right\}\end{aligned} \tag{49}$$

Which is an implicit (stable) time differencing scheme. However, a history of solution values (vectors) must be kept dating back three time steps. In addition, some type of starting scheme must be developed to obtain this solution history (I use Crank-Nicholson with very small time steps, extract several solutions, store and use these solutions)

Work in Progress (Adams-Bashforth)

If we add a term to our time expansion term and require the implicit β_0 be zero, we can develop a higher order explicit Adams-Bashforth scheme. The

m	β_0	β_1	β_2	β_3	β_4	Truncation
1	0	1	0	0	0	first
2	0	$\frac{3}{2}$	$-\frac{1}{2}$	0	0	second
3	0	$\frac{23}{12}$	$-\frac{16}{12}$	$\frac{5}{12}$	0	third
4	0	$\frac{55}{24}$	$-\frac{59}{12}$	$\frac{37}{24}$	$-\frac{9}{24}$	fourth

An explicit representation of Stokes equations become,

$$\begin{aligned}[\mathcal{M}]\{\mathcal{U}\}^{n+1} &= \left[[\mathcal{M}]-\beta_1\Delta t[\widetilde{\mathcal{K}}]\right]\{\mathcal{U}\}^{n} \\ &-\beta_2\Delta t[\widetilde{\mathcal{K}}]\{\mathcal{U}\}^{n-1}-\beta_3\Delta t[\widetilde{\mathcal{K}}]\{\mathcal{U}\}^{n-2}-\beta_4\Delta t[\widetilde{\mathcal{K}}]\{\mathcal{U}\}^{n-3} \\ &+\Delta t\left\{\beta_0\{\mathcal{F}\}^{n+1}+\beta_1\{\mathcal{F}\}^{n}+\beta_2\{\mathcal{F}\}^{n-1}+\beta_3\{\mathcal{F}\}^{n-2}+\beta_4\{\mathcal{F}\}^{n-3}\right\}\end{aligned} \tag{50}$$

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Since the Mass matrix may be rendered diagonal by the use of Chebyshev-Gauss-Labatto integration no inversion of a mass matrix is required! If we solve this system for a predictor step,

$$\begin{aligned}\{\mathcal{U}\}^P = &[\mathcal{M}]^{-1}\left[[\mathcal{M}] - \beta_1\Delta t[\widetilde{\mathcal{K}}]\right]\{\mathcal{U}\}^n \\ &- [\mathcal{M}]^{-1}\left\{\beta_2\Delta t[\widetilde{\mathcal{K}}]\{\mathcal{U}\}^{n-1} + \beta_3\Delta t[\widetilde{\mathcal{K}}]\{\mathcal{U}\}^{n-2} + \beta_4\Delta t[\widetilde{\mathcal{K}}]\{\mathcal{U}\}^{n-3}\right\} \\ &+\Delta t\,[\mathcal{M}]^{-1}\left\{\beta_0\{\mathcal{F}\}^{n+1} + \beta_1\{\mathcal{F}\}^n + \beta_2\{\mathcal{F}\}^{n-1} + \beta_3\{\mathcal{F}\}^{n-2} + \beta_4\{\mathcal{F}\}^{n-3}\right\}\end{aligned} \tag{51}$$

We now take this prediction of the velocities at the present step and insert its value into the implicit Adams-Moulton differencing equation,

$$\begin{aligned}[\mathcal{M}]\{\mathcal{U}\}^{n+1} = &\left[[\mathcal{M}] - \beta_1\Delta t[\widetilde{\mathcal{K}}]\right]\{\mathcal{U}\}^n - \beta_0\Delta t[\widetilde{\mathcal{K}}]\{\mathcal{U}\}^P \\ &-\beta_2\Delta t[\widetilde{\mathcal{K}}]\{\mathcal{U}\}^{n-1} - \beta_3\Delta t[\widetilde{\mathcal{K}}]\{\mathcal{U}\}^{n-2} \\ &+\Delta t\left\{\beta_0\{\mathcal{F}\}^{n+1} + \beta_1\{\mathcal{F}\}^n + \beta_2\{\mathcal{F}\}^{n-1} + \beta_3\{\mathcal{F}\}^{n-2}\right\}\end{aligned} \tag{52}$$

As a result, we never assemble nor invert our stiffness matrix. However, there does exist a tight restriction on the time step resulting from the eigenvalue structure of the stiffness matrix. Since a penalty term has been used the eigenvalue structure is *shifted* towards a singular matrix.

At present, the code uses the fully implicit formulation.

This may change within the next six months.

End of (Work in Progress)

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2. Navier-Stokes equations

Inclusion of the convection terms in Stokes equations will, by definition, result in Navier-Stokes equations. If we rewrote our Stokes equations 45 to include the nonlinear advection terms we have,

$$[\mathcal{M}]\left(\dot{\mathcal{U}}\right) + \left[\widetilde{\mathcal{K}}\right](\mathcal{U}) + [\mathcal{N}(u)](\mathcal{U}) = (\mathcal{F}) \tag{53}$$

Where, $[\mathcal{N}(u)]$, result from the convection terms. We now integrate our equation over a representative time step,

$$(\mathcal{U})^{n+1} - (\mathcal{U})^{n} = \int_{t^n}^{t^{n+1}} \left\{(\mathcal{F}) - \left[\widetilde{\mathcal{K}}\right](\mathcal{U})\right\} dt - \int_{t^n}^{t^{n+1}} [\mathcal{N}(u)](\mathcal{U})\, dt \tag{54}$$

The first integral term is the Stokes portion of the Navier-Stokes equation and has already been dealt with. The second integral term on the right hand side is the term we will focus on in this section.

2.1. Runge-Kutta Method

At this point it may be instructive to look closely at a second order Runge-Kutta method. A second order variety of these methods is shown below,

$$\begin{aligned} u_i^{(1)} &= \Delta t \mathcal{F}_i\left(t, u_j\right) \\ u_i^{(2)} &= \Delta t \mathcal{F}_i\left(t + \Delta t/2, u_j + u_j^{(1)}/2\right) \\ u_i^{n+1} &= u_i^n + u_i^{(2)} \end{aligned} \tag{55}$$

Where in our case,

$$\mathcal{F}_k\left(t, u_j\right) = \frac{\partial u_k}{\partial t} = -u_i \frac{\partial u_k}{\partial x_i} \tag{56}$$



For two dimensions we have,

$$\begin{aligned} u^{(1)} &= -\Delta t \left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \right] \\ v^{(1)} &= -\Delta t \left[u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} \right] \end{aligned} \tag{57}$$

Taking a standard weighted residual approach we multiply these equations by a weighting function, our basis functions, and integrate over the domain of our problem.

$$\begin{aligned} \int_{\mathcal{D}} \psi_k u^{(1)} d\mathcal{D} &= -\Delta t \int_{\mathcal{D}} \psi_k \left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \right] d\mathcal{D} \\ \int_{\mathcal{D}} \psi_k v^{(1)} d\mathcal{D} &= -\Delta t \int_{\mathcal{D}} \psi_k \left[u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} \right] d\mathcal{D} \end{aligned} \tag{58}$$

There exist only three types of matrices in these two equations. These matrices are,

$$\begin{aligned} M_{ij} &= \int_{\mathcal{D}} \psi_i \psi_j d\mathcal{D} \\ C^x_{ijk} &= \int_{\mathcal{D}} \psi_i \psi_j \frac{\partial \psi_k}{\partial x} d\mathcal{D} \\ C^y_{ijk} &= \int_{\mathcal{D}} \psi_i \psi_j \frac{\partial \psi_k}{\partial y} d\mathcal{D} \end{aligned} \tag{59}$$

Writing our first step we have,

$$\begin{aligned} M_{ij} u^{(1)}_j &= -\Delta t \left[C^x_{ijk} u^n_j u^n_k + C^y_{ijk} v^n_j u^n_k \right] \\ M_{ij} v^{(1)}_j &= -\Delta t \left[C^x_{ijk} u^n_j v^n_k + C^y_{ijk} v^n_j v^n_k \right] \end{aligned} \tag{60}$$

Using the following,

$$\begin{aligned} \tilde{u}_n &= u_n + \frac{u^{(1)}_n}{2} \\ \tilde{v}_n &= v_n + \frac{v^{(1)}_n}{2} \end{aligned} \tag{61}$$

Our Second step may be written as,

$$\begin{aligned} \mathcal{F}_x^{rk} &= M_{ij} u_j^{(2)} = -\Delta t \left[C_{ijk}^x \tilde{u}_j \tilde{u}_k + C_{ijk}^y \tilde{v}_j \tilde{u}_k \right] \\ \mathcal{F}_y^{rk} &= M_{ij} v_j^{(2)} = -\Delta t \left[C_{ijk}^x \tilde{u}_j \tilde{v}_k + C_{ijk}^y \tilde{v}_j \tilde{v}_k \right] \end{aligned} \tag{62}$$

Once these terms $u_j^{(2)}$ and $v_j^{(2)}$ are determined we may use the last equation in the Runge-Kutta set to obtain our velocities at the next time step.

$$\begin{aligned} u_j^{n+1} - u_j^n &= u_j^{(2)} \\ v_j^{n+1} - v_j^n &= v_j^{(2)} \end{aligned} \tag{63}$$

Careful observation of these last equations and our Stokes equations, 47, we see the only addition to our equations is the appearance of an added right hand side vector, 62.

Hence for second order accurate time dependent simulation of Navier-Stokes equations we have,

$$\left[[\mathcal{M}] + \frac{\Delta t}{2} [\widetilde{\mathcal{K}}] \right] \{\mathcal{U}\}^{n+1} = \left[[\mathcal{M}] - \frac{\Delta t}{2} [\widetilde{\mathcal{K}}] \right] \{\mathcal{U}\}^n + \frac{\Delta t}{2} \left\{ \{\mathcal{F}\}^n + \{\mathcal{F}\}^{n+1} \right\} + \{\mathcal{F}\}^{rk} \tag{64}$$

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3. External Libraries

- TAUCS– C library of matrix solvers. Developed by Professor Sivan Toledo.
  - Used by Mathematica as their sparse solver.
  - Uses compact column storage for non zero sparse matrix elements.
  - Uses out-of-core, left-looking supernodal sparse Cholesky Factorization.
  - Uses Multifrontal Supernodal Cholesky Factorization.
  - Uses several reordering codes and Interfaces to existing ordering codes.
  - Contains preconditioned conjugate gradient solvers.
  - Contains several preconditioner.
  - ETC.
- Cilk– a language for multithreaded parallel programming based on ANSI C. Used by TAUCS, Not completely implemented YET.
- UMFPACK– C library of matrix solvers. Developed by Tim Davis.
  - Used by Matlab as their sparse solver.
  - Uses compact column storage for non zero sparse matrix elements.
  - Uses Multifrontal Supernodal Cholesky Factorization.
  - Uses AMD, a set of routines for ordering a sparse matrix prior to Cholesky factorization.
  - ETC.
- BLAS– Basic Linear Algebra Subroutines

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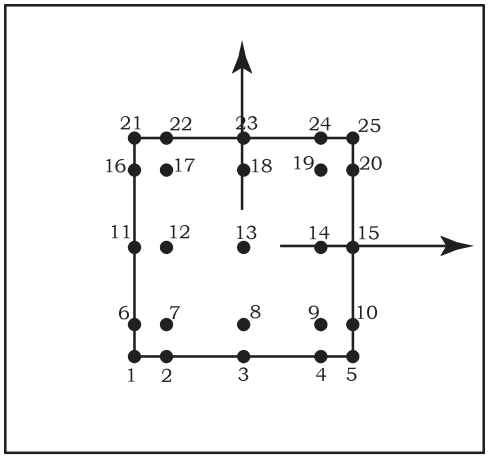
Flow Examples

Unsteady flow development around a cylinder (Approximate Factorization) (Re=100)
Unsteady flow development around a cylinder (Approximate Factorization) (Re=100)
Unsteady flow around a cylinder (Approximate Factorization) (Re=100)
Unsteady flow around a cylinder (Approximate Factorization) (Re=100)
Attempt to blow the unsteady flow by velocity equal to main stream flow (Penalty) (Re=100)
Attempt to blow the unsteady flow by velocity equal to main stream flow (Penalty) (Re=100)
Attempt to blow the unsteady flow by velocity 3 times main stream flow (Penalty) (Re=100)
Attempt to blow the unsteady flow by velocity 3 times main stream flow (Penalty) (Re=100)

5. Spectral-vs-Finite Elements

Actually there is no difference between the two except...

- Nodes in Spectral Elements are placed at the Gauss-Labatto points, see next page
- Integration is carried out in 1-D and results in two or three dimensions are obtained by a tensor product of the 1-D results.



$$
\begin{aligned}
M_{hg} &= \int_{-1}^{+1}\int_{-1}^{+1} \psi_h \psi_g dr ds \\
&= \int_{-1}^{+1}\int_{-1}^{+1} \varphi_i(r)\,\varphi_j(s)\,\varphi_k(r)\,\varphi_l(s)\,dr ds \\
&= \left[\int_{-1}^{+1} \varphi_i(r)\,\varphi_k(r)\,dr\right]\left[\int_{-1}^{+1} \varphi_j(s)\,\varphi_l(s)\,ds\right] \\
M_{hg} &= \widetilde{\mathcal{M}}_{ik}\widetilde{\mathcal{M}}_{jl}
\end{aligned}
\tag{65}
$$

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6. Works in Progress

6.1. Arbitrary point-line-surface constraints

Any matrix may be constrained by a penalty or Lagrange multiplier constraint. Given,

$$[K]\{x\} = \{b\} \tag{66}$$

subject to a constraint equation,

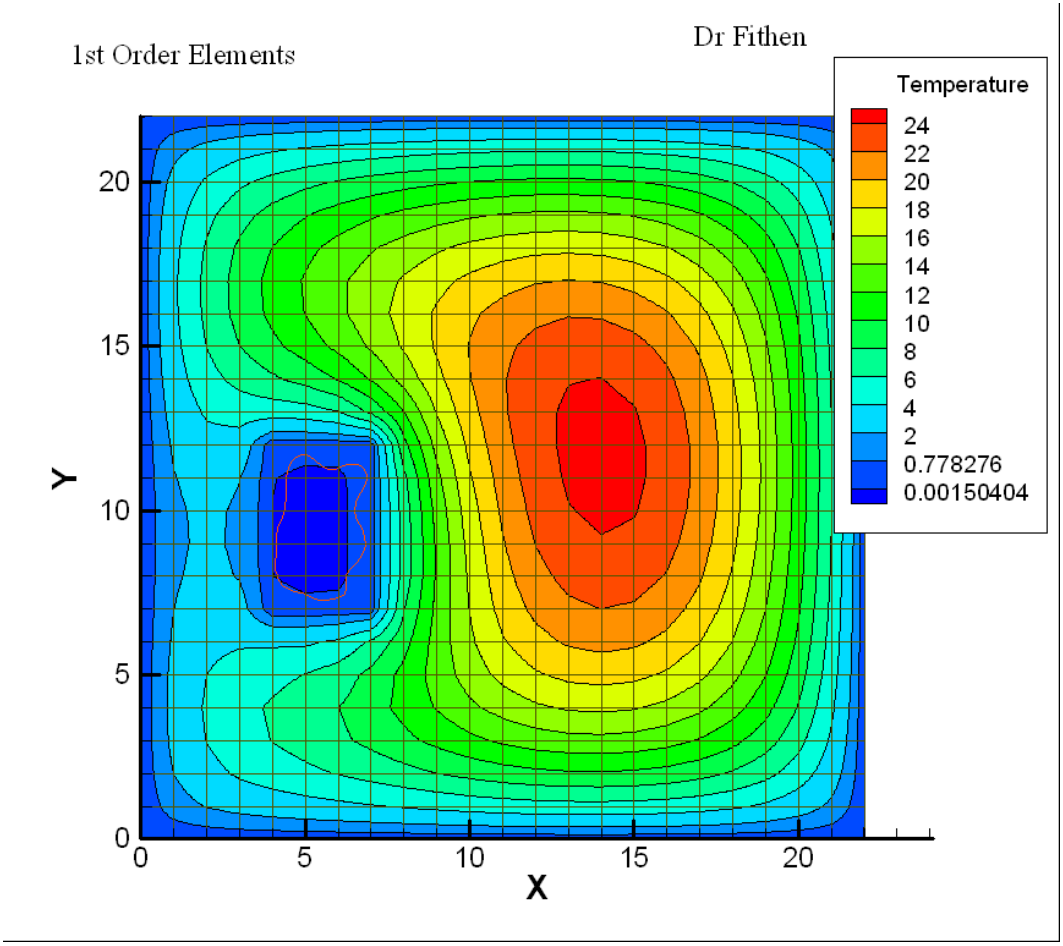
$$[C]\{x\} = \{c\} \tag{67}$$

The Lagrange multiplier constrain of this matrix system gives,

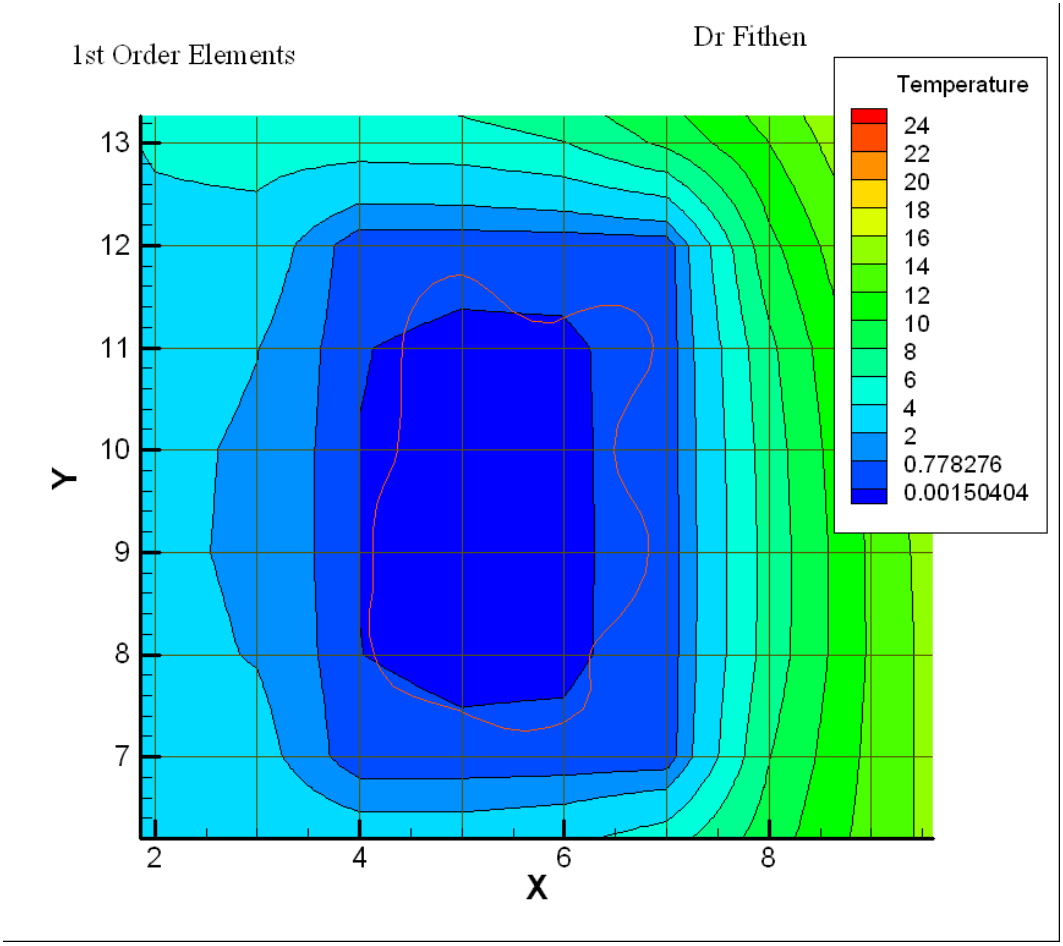
$$\begin{bmatrix} [K] & [C]^T \\ [C] & [0] \end{bmatrix} \begin{Bmatrix} \{x\} \\ \{\lambda\} \end{Bmatrix} = \begin{Bmatrix} \{b\} \\ \{c\} \end{Bmatrix} \tag{68}$$

Using this type of system along with line integrals through 2-D elements we obtain....

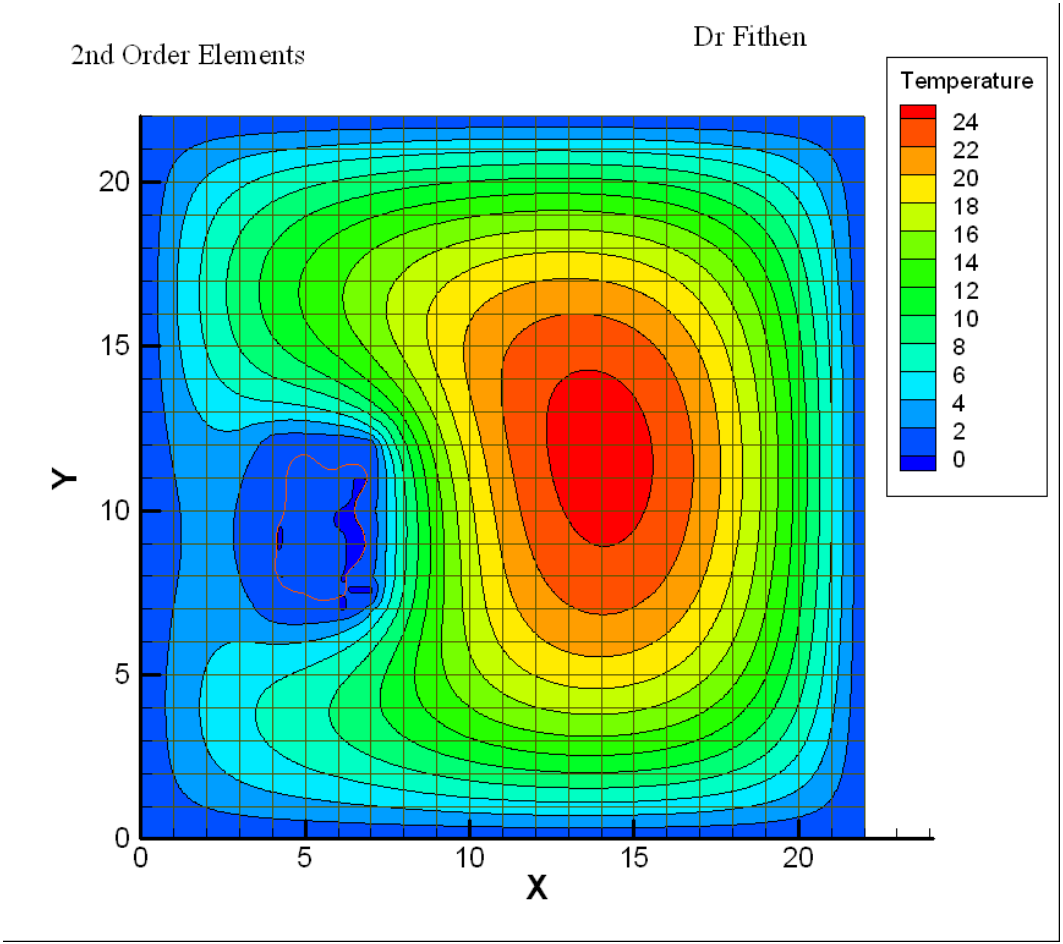
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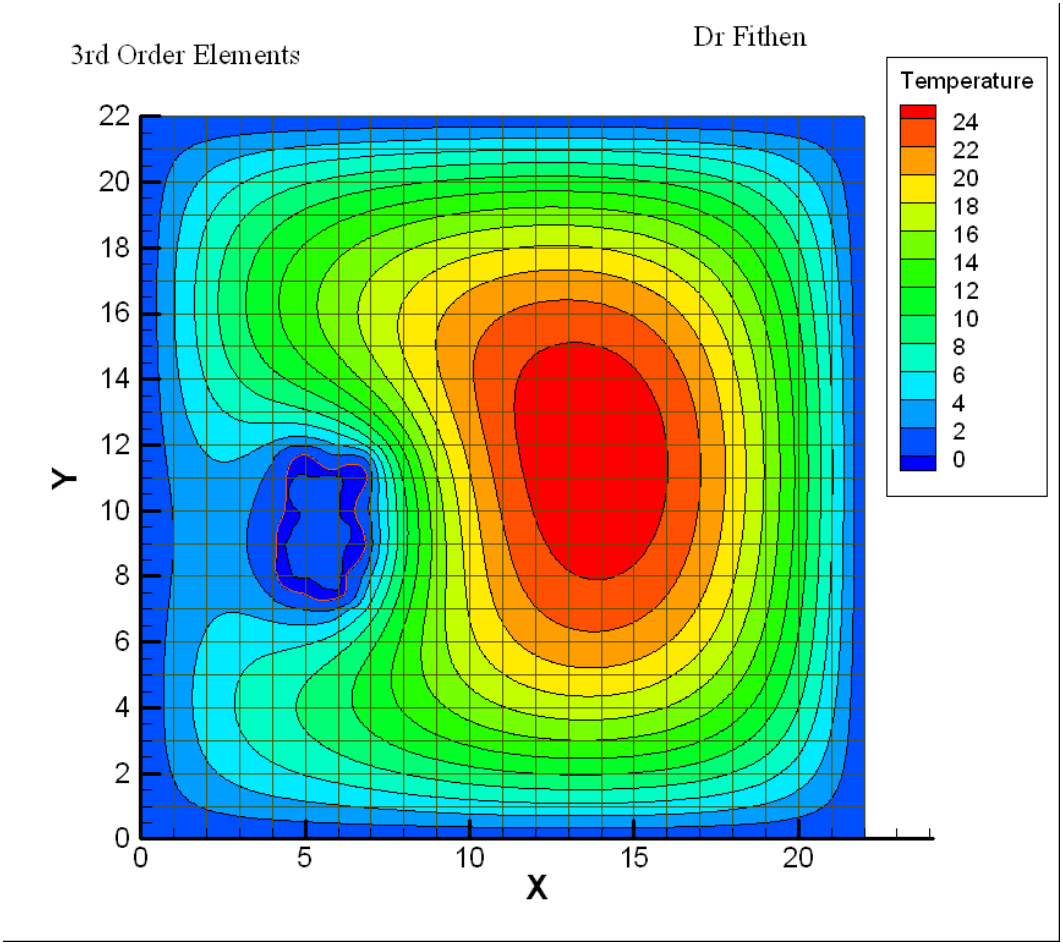
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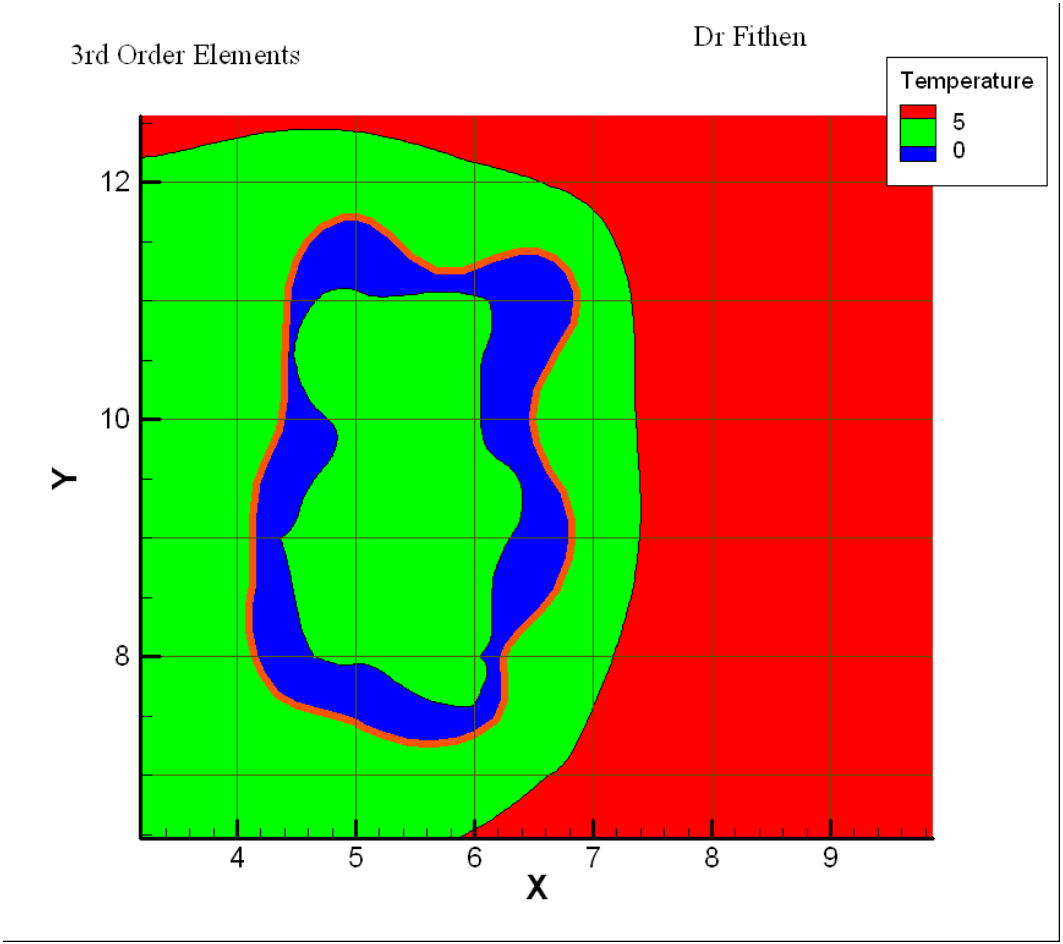
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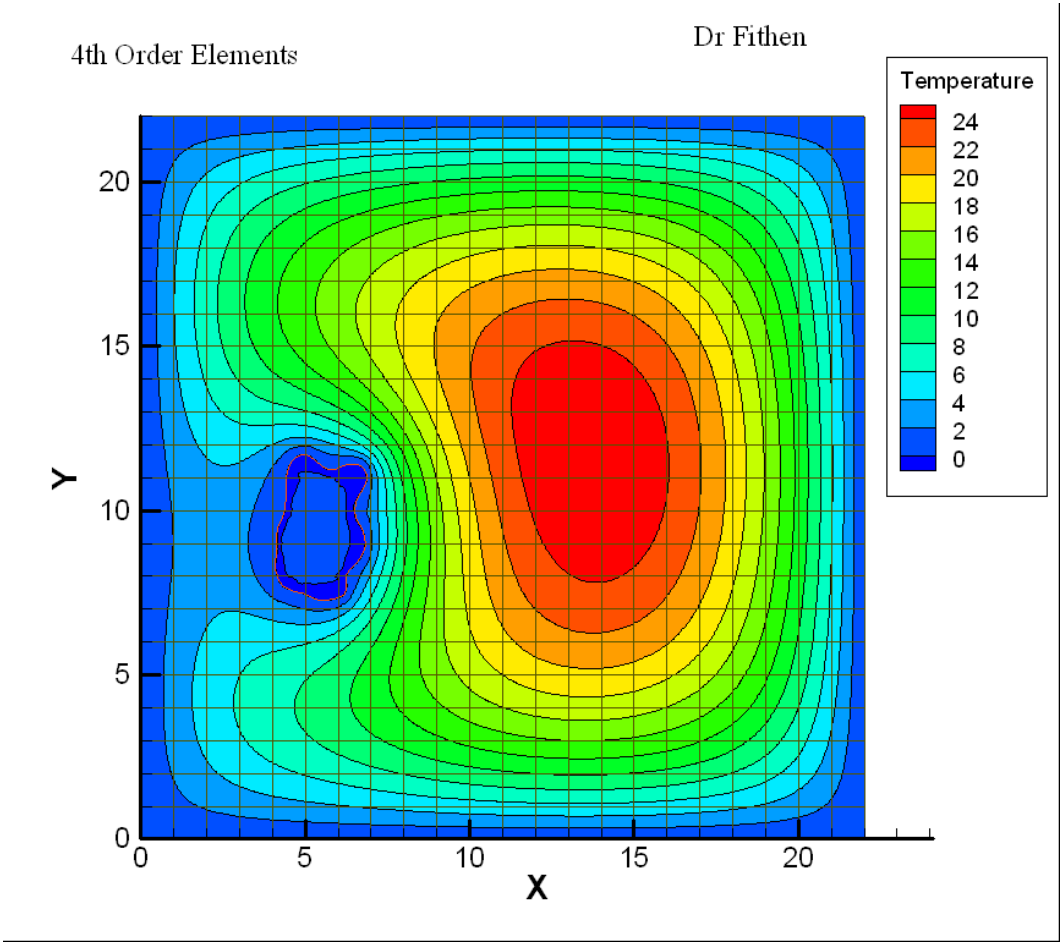
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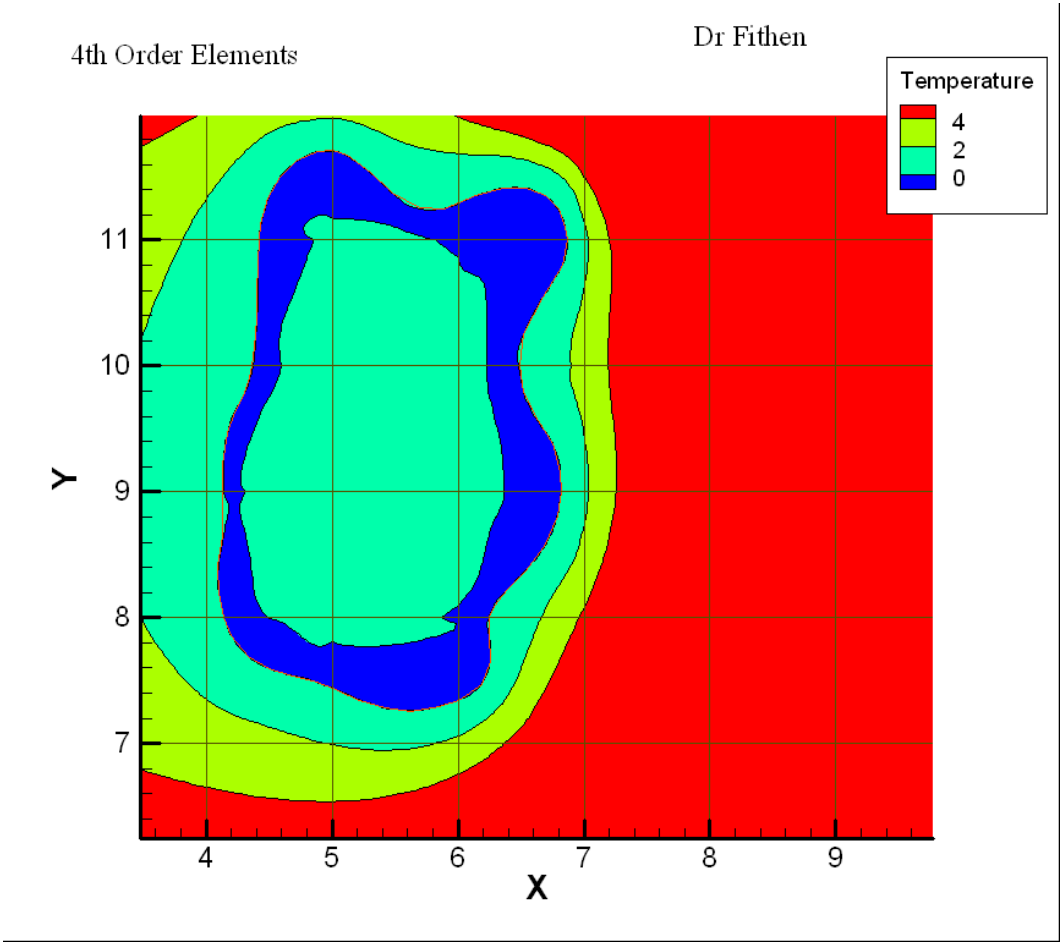
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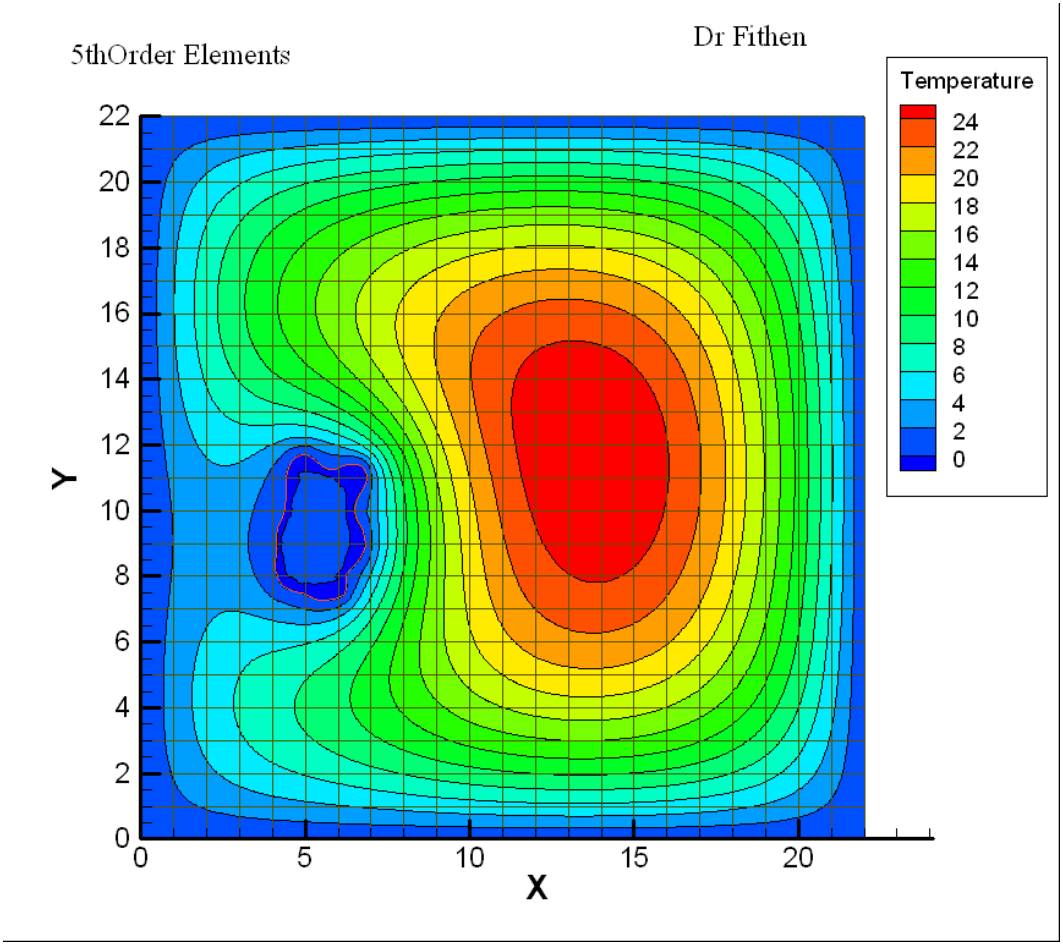
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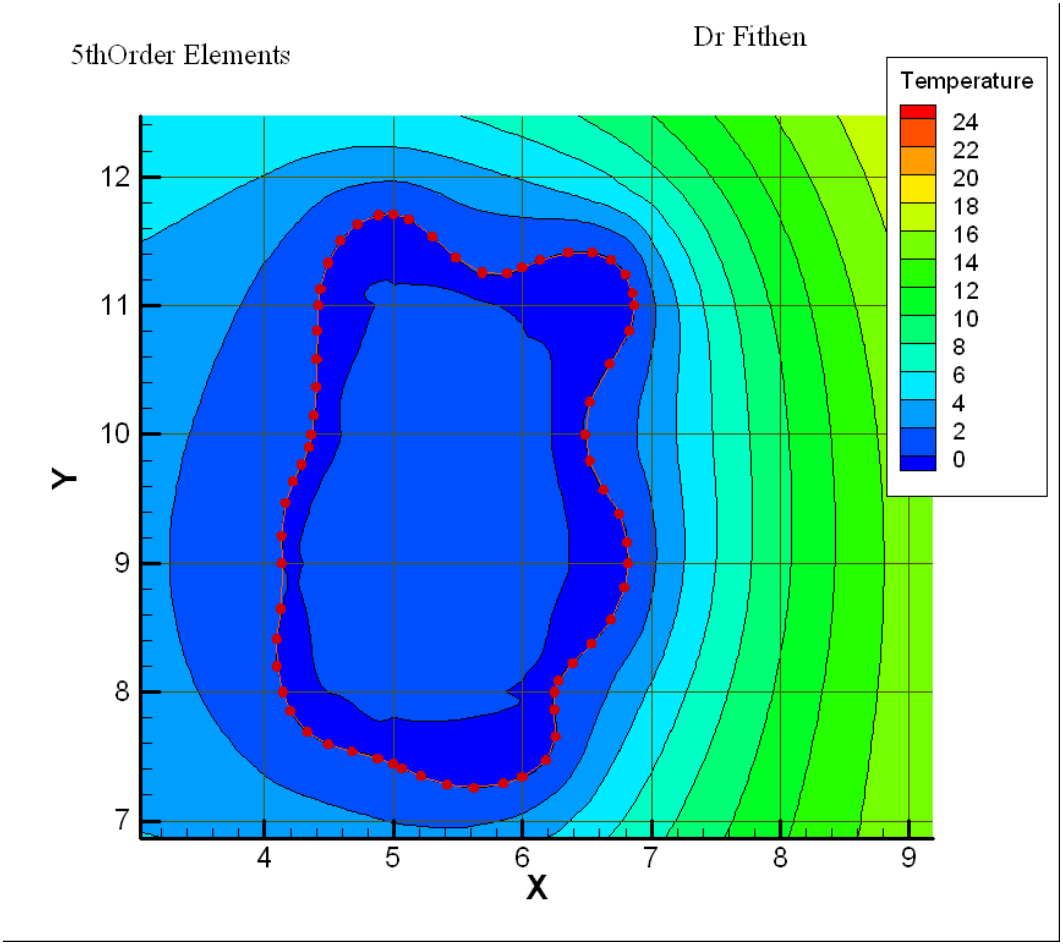
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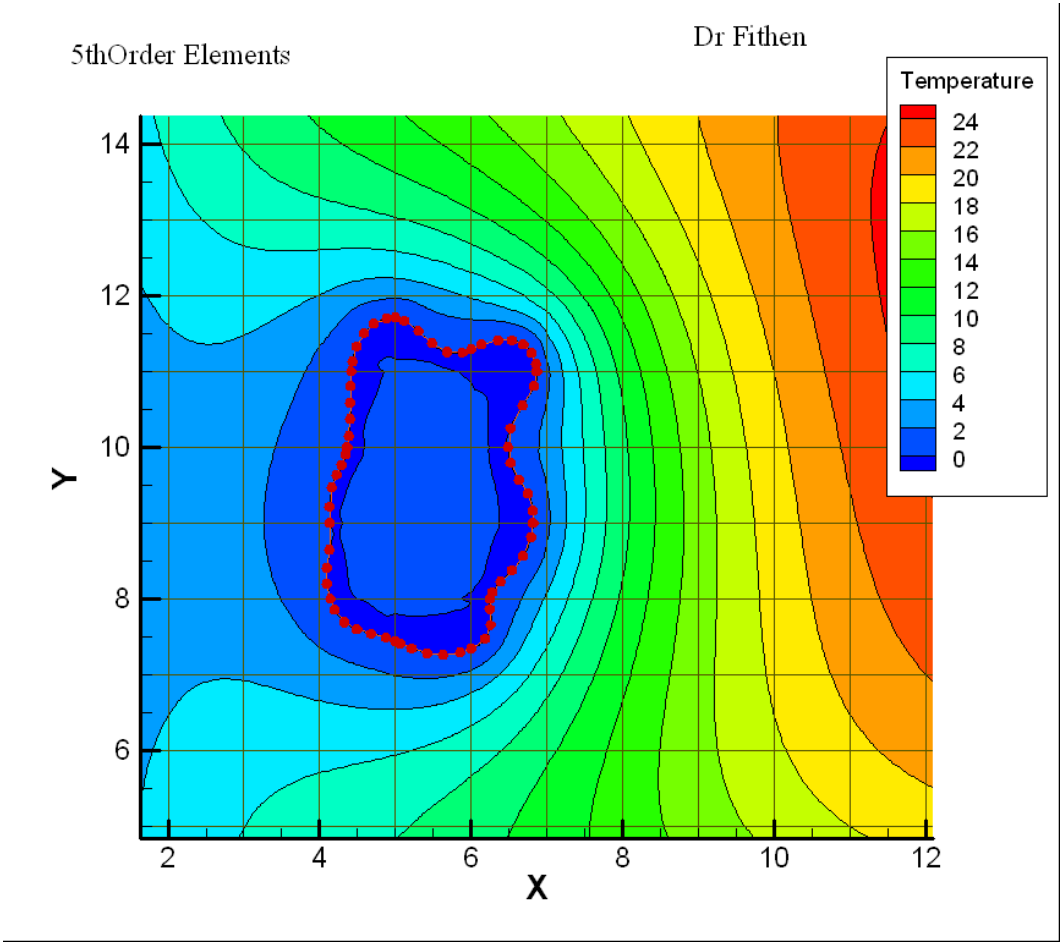
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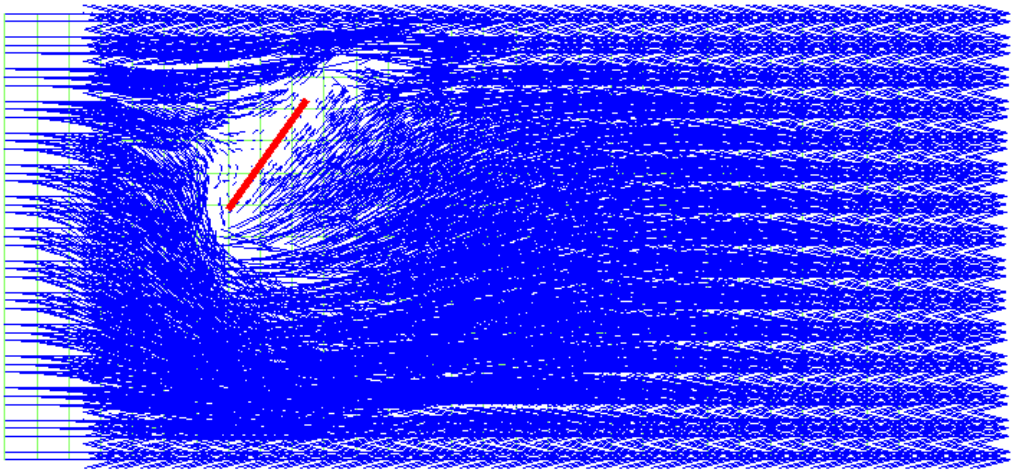
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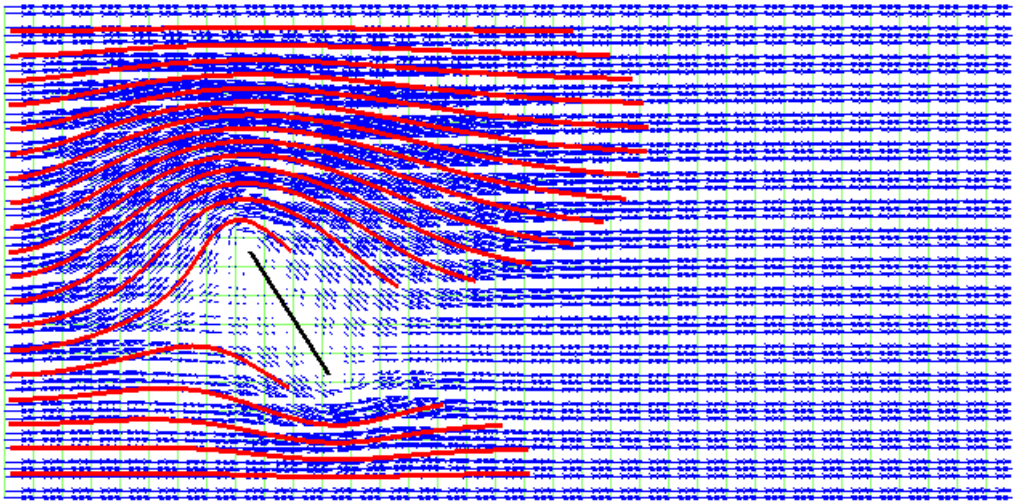


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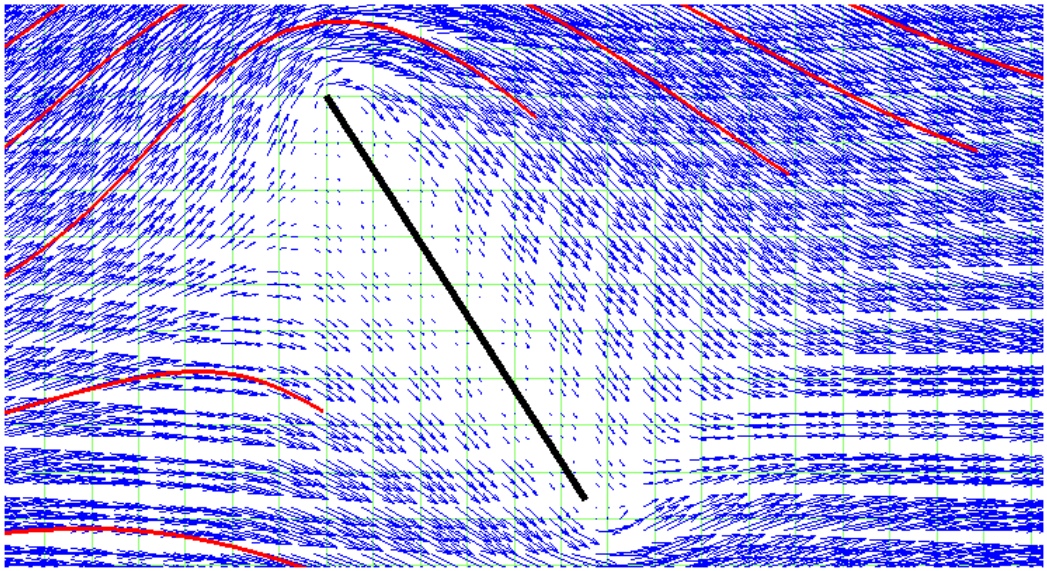
- 3rd Order Penalty Stokes Flow Test problem
- Coarse Grid.
- Lagrange Multiplier applied along arbitrary line.

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- 3rd Order Penalty Stokes Flow Test problem
- Medium resolution Grid.
- Lagrange Multiplier applied along arbitrary line.

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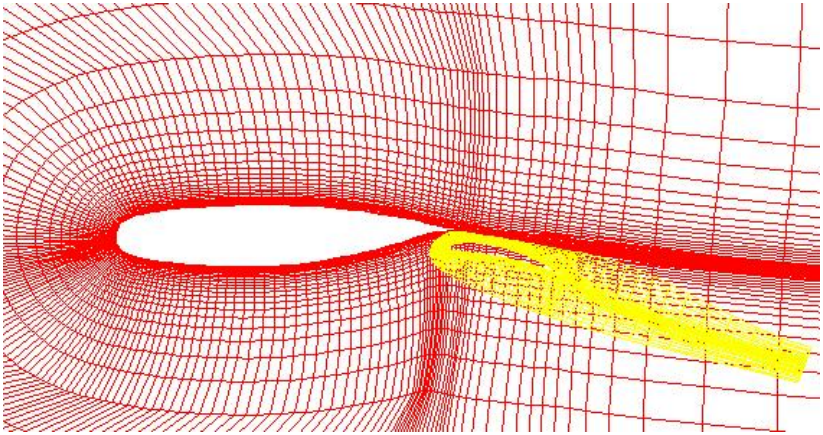


- 3^{rd} Order Penalty Stokes Flow Test problem
- Medium resolution Grid.
- Lagrange Multiplier applied along arbitrary line.

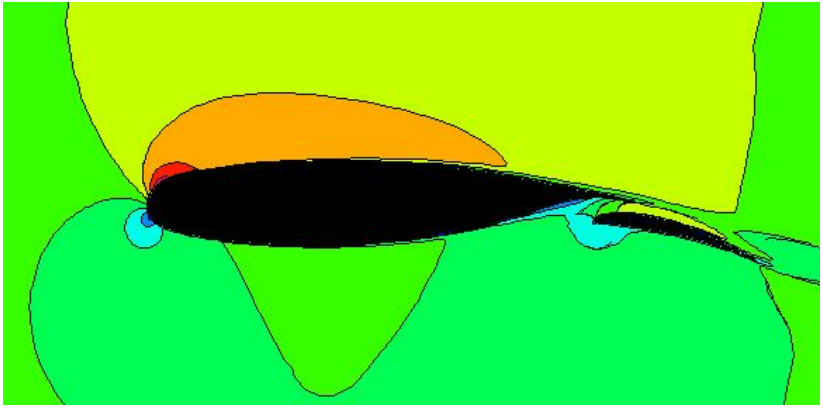
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6.2. Chimera Overset grid method

Typical example of overset grid...



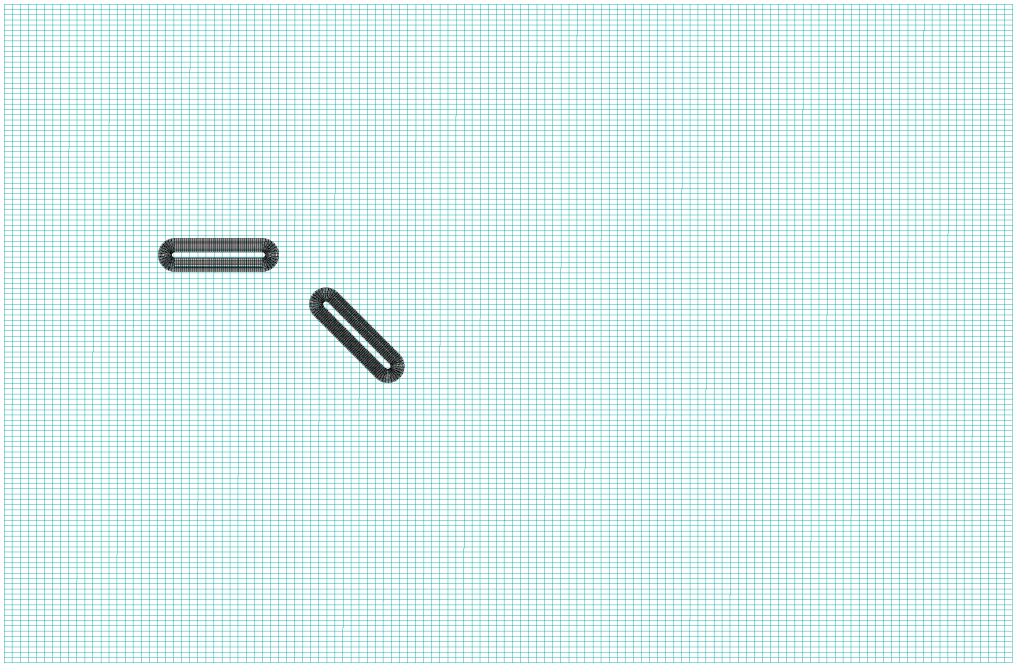
Control Volume solution overset grid (iterative/bilinear interpolation)...



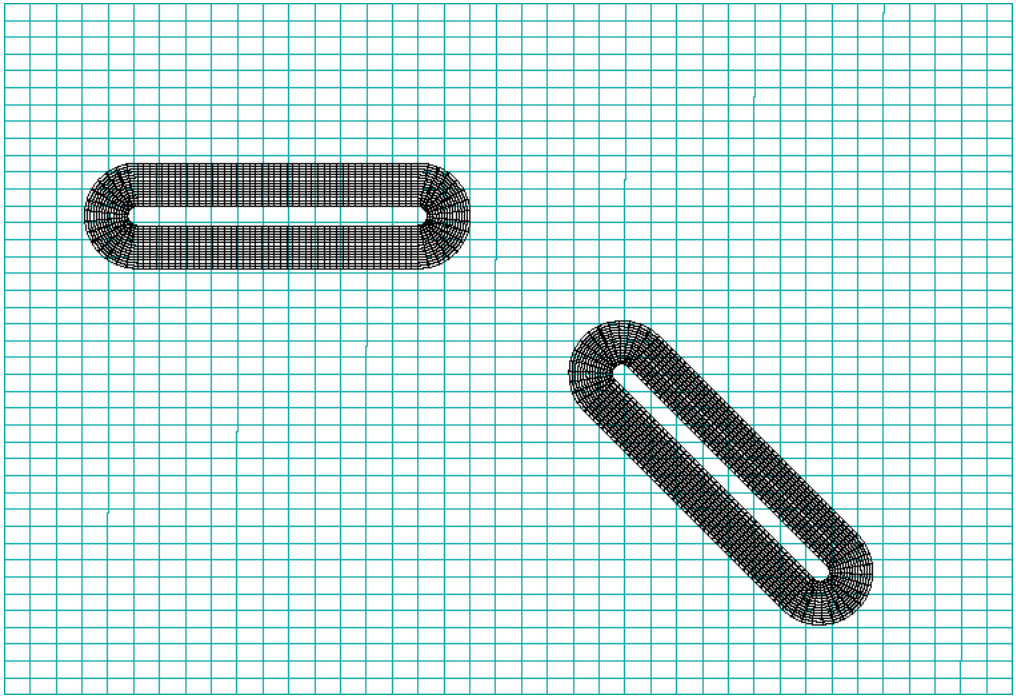
For spectral/finite elements a direct constraint may be established via Lagrange multipliers for any variable as,

$$\begin{bmatrix} [K_1] & [0] & [0] & [C_1]^T \\ [0] & [K_2] & [0] & [C_2]^T \\ [0] & [0] & [K_3] & [C_3]^T \\ [C_1] & [C_2] & [C_3] & [0] \end{bmatrix} \begin{Bmatrix} \{x_1\} \\ \{x_2\} \\ \{x_3\} \\ \{\lambda\} \end{Bmatrix} = \begin{Bmatrix} \{b_1\} \\ \{b_2\} \\ \{b_3\} \\ \{c\} \end{Bmatrix} \tag{69}$$

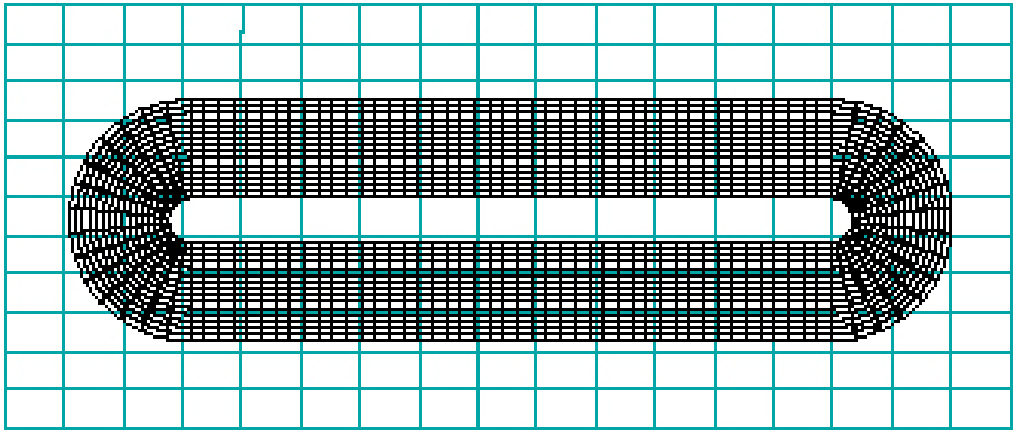
Example using NSFIT2D penalty version...
3 Grids, 1 background and 2 overset



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Penalty Navier-Stokes simulation (Re=1000)

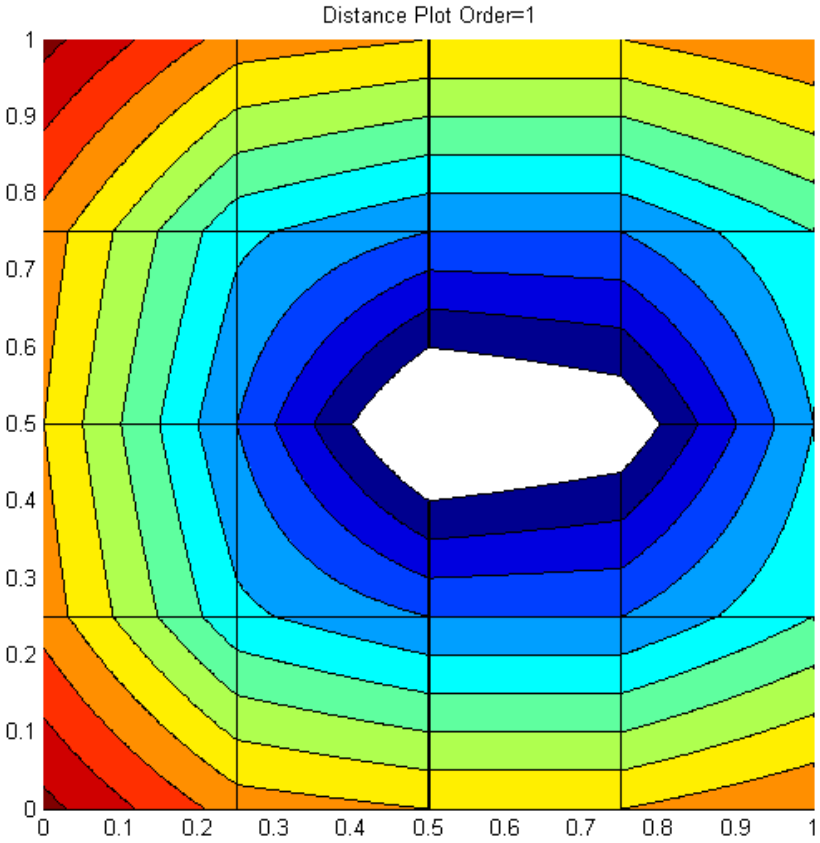
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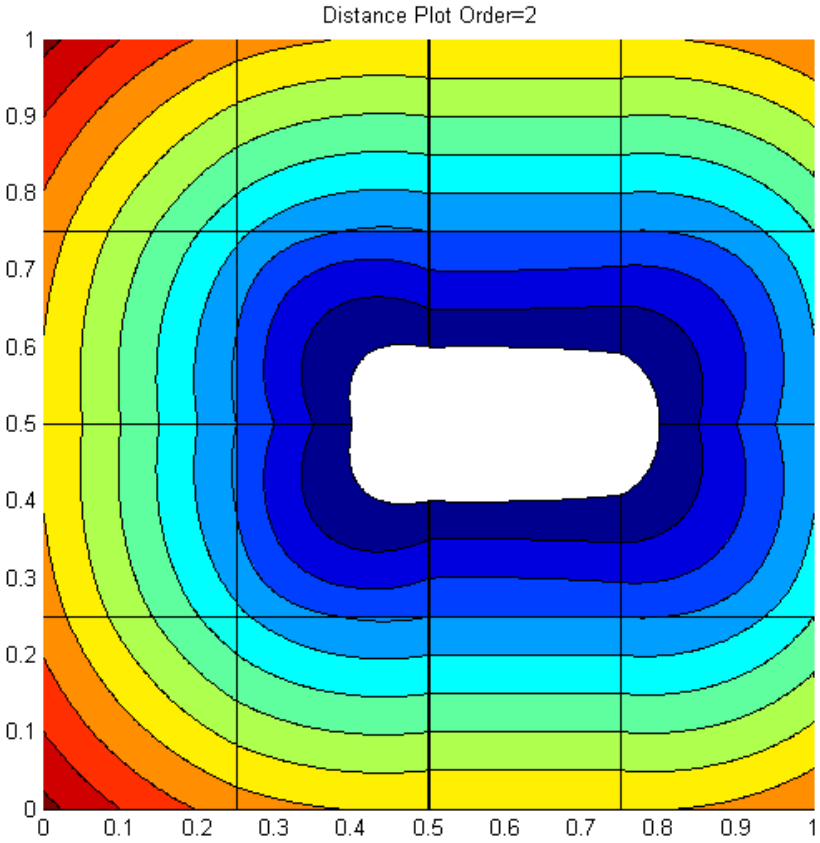
I primarily use distance functions to determine constraint locations. In fact, these distance functions can be used themselves as arbitrary constraint points.

Example..... Next Page

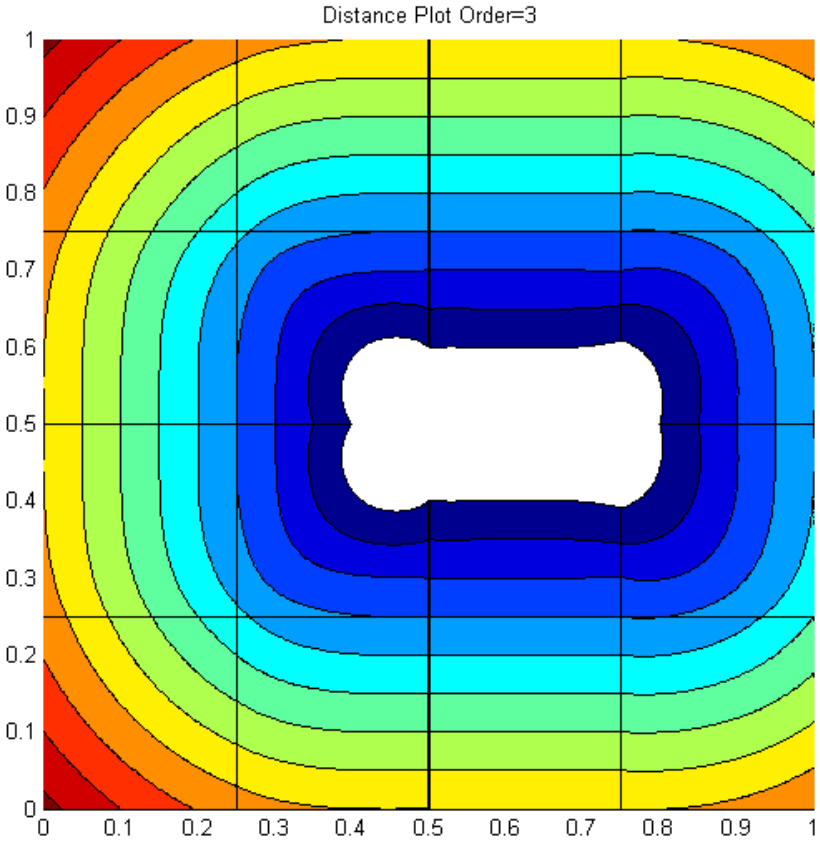
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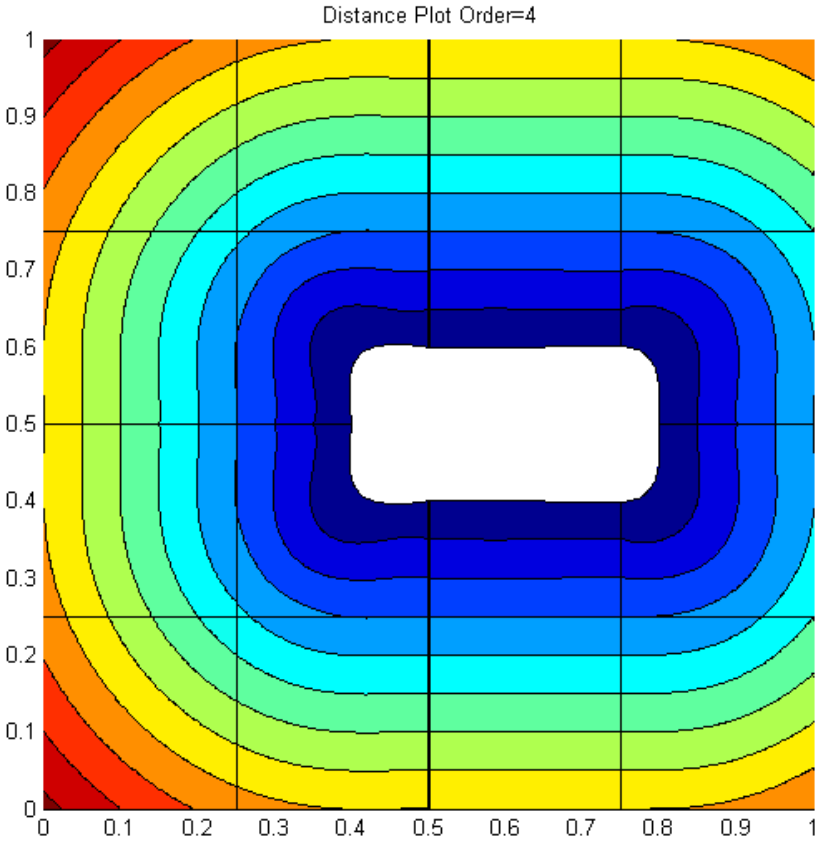
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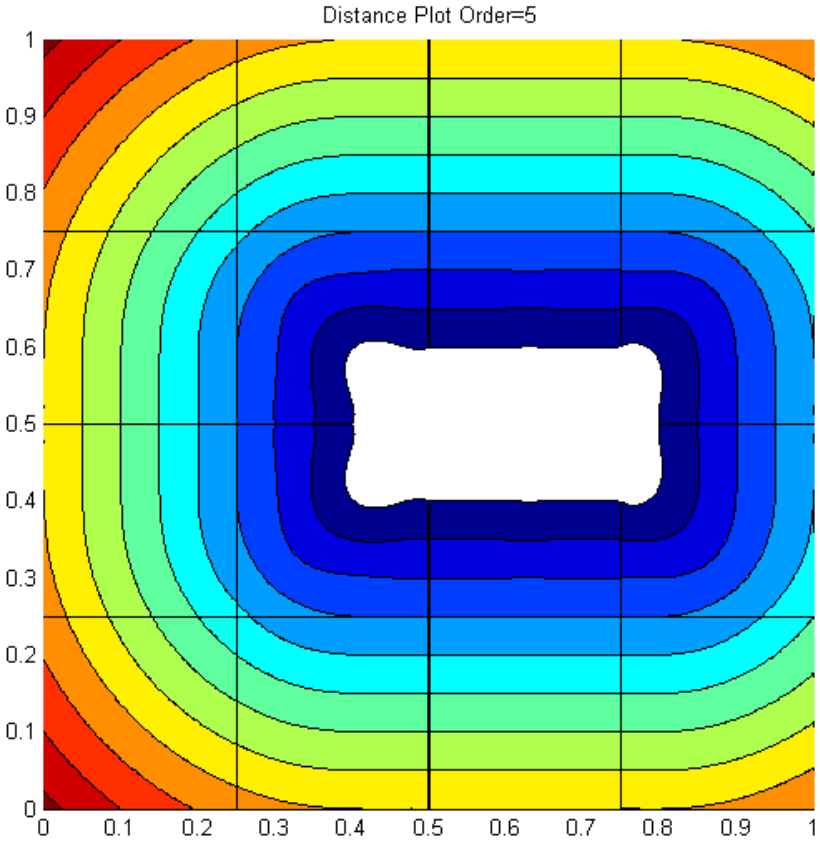
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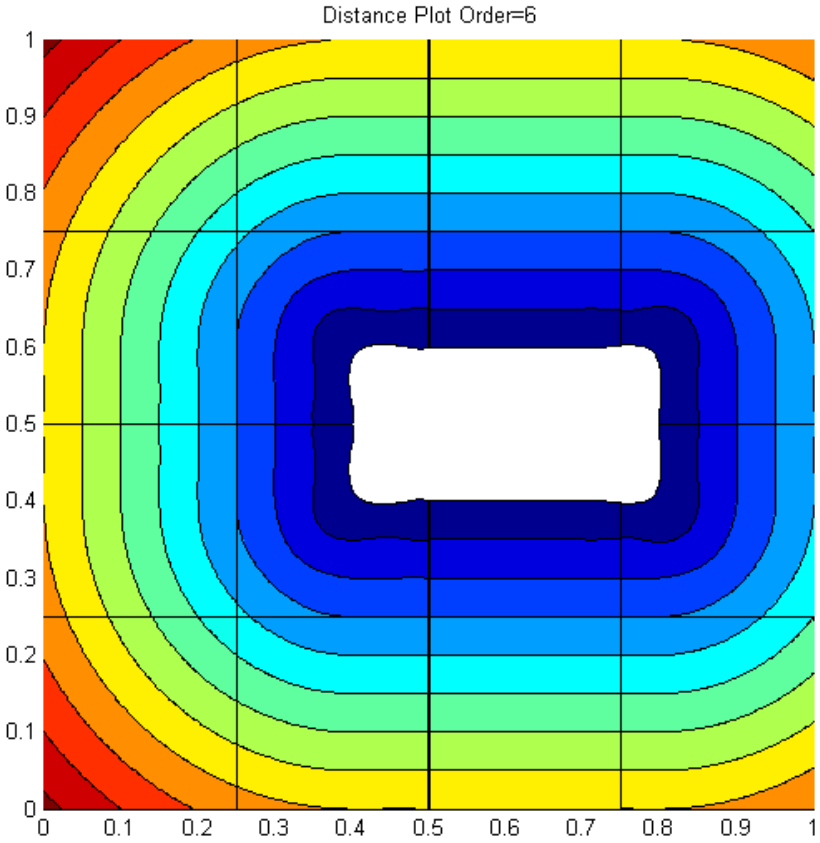
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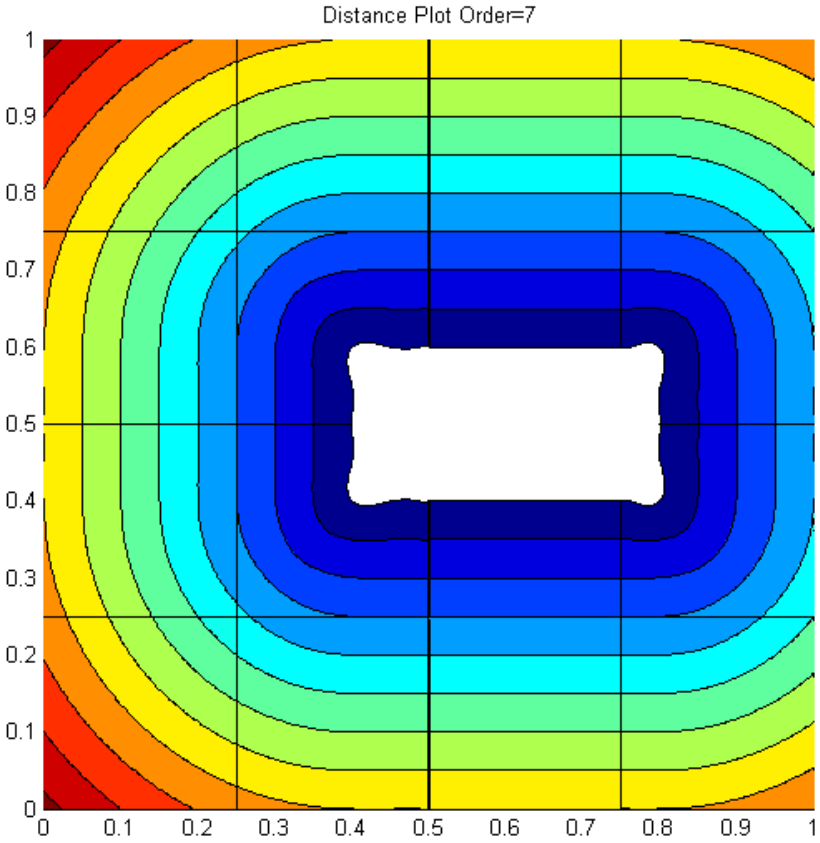
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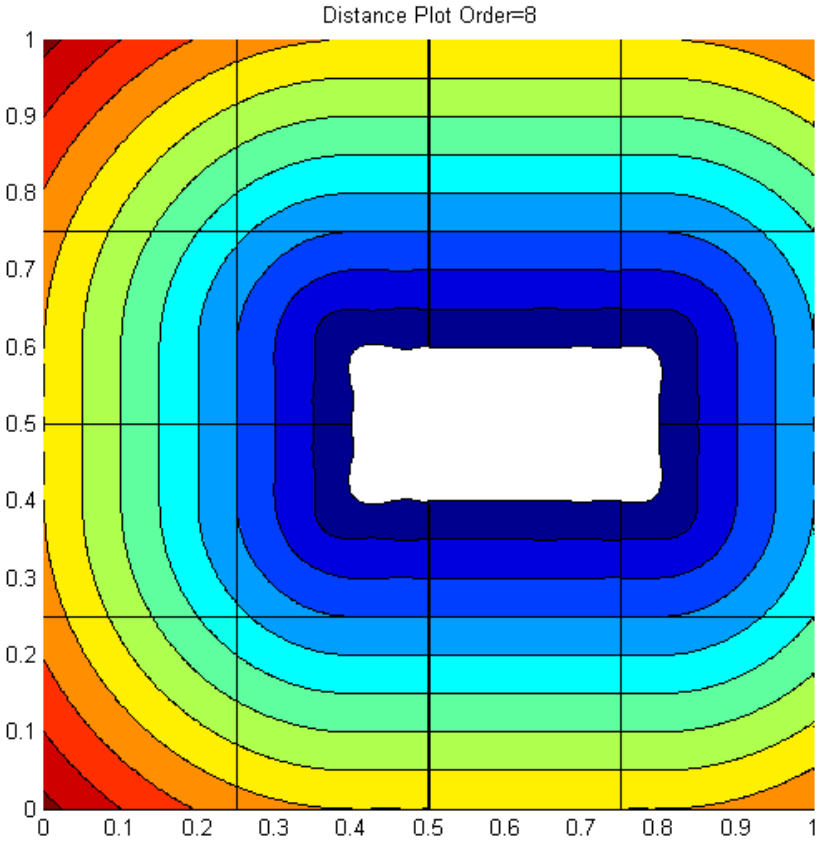
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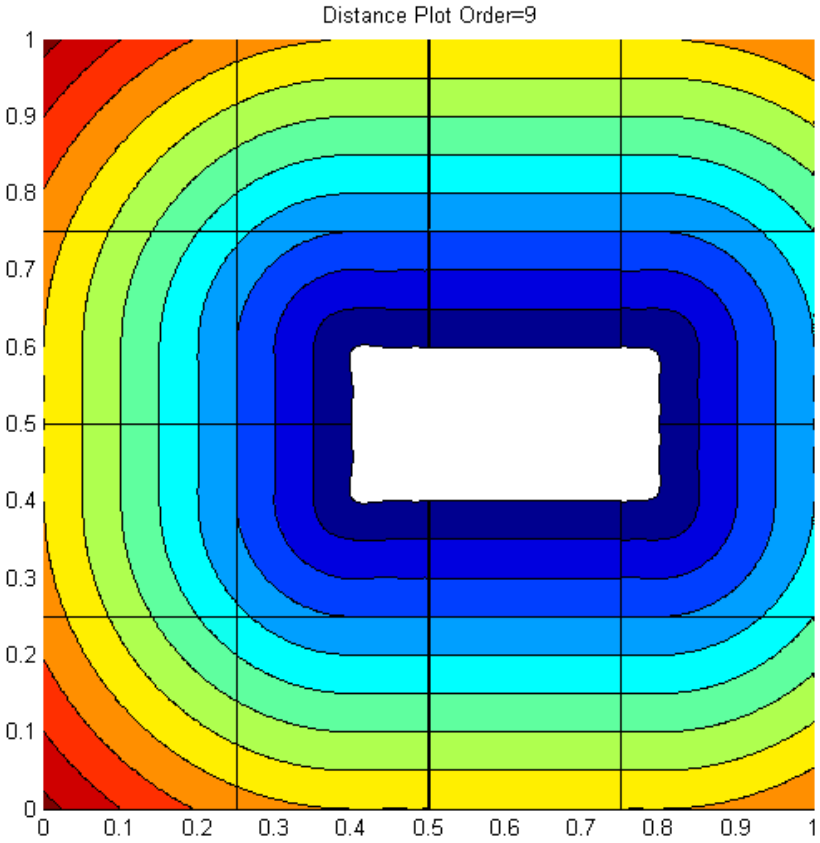
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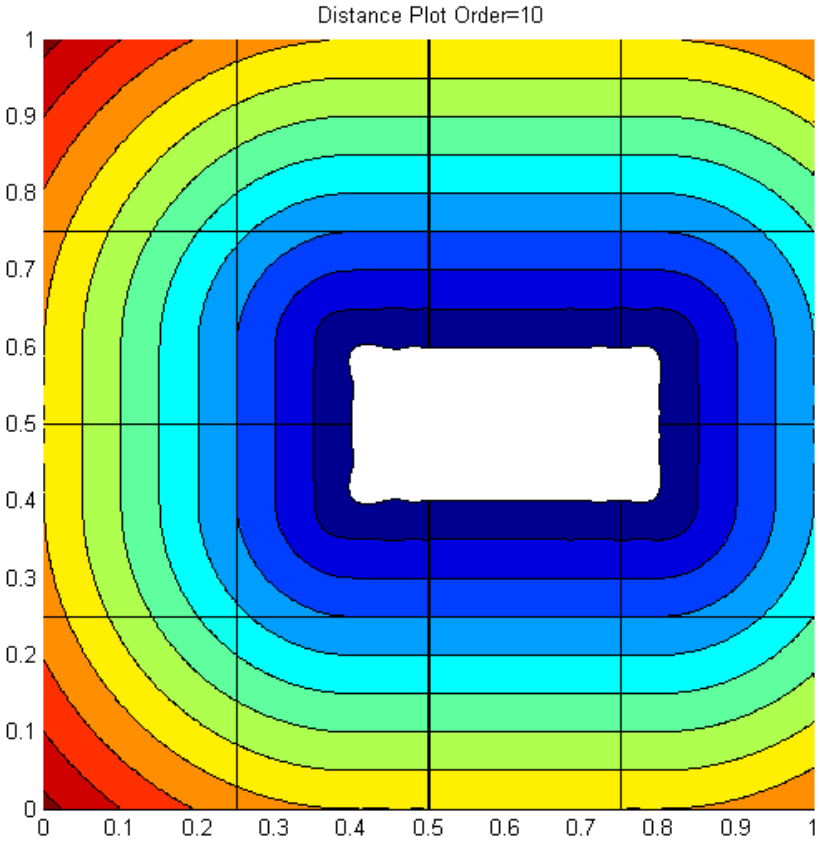
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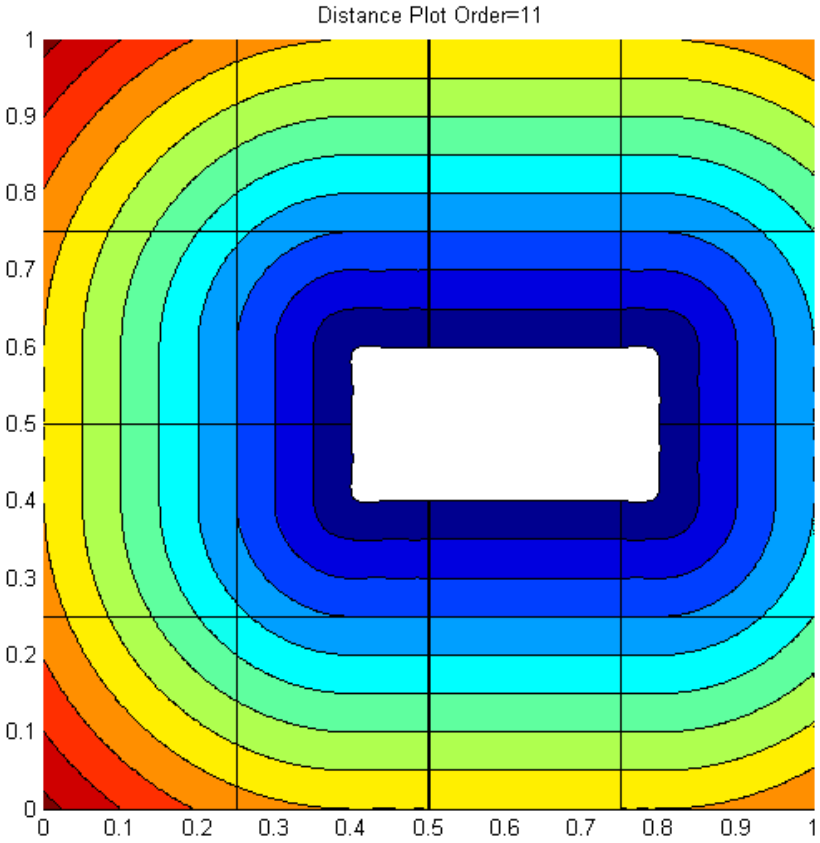
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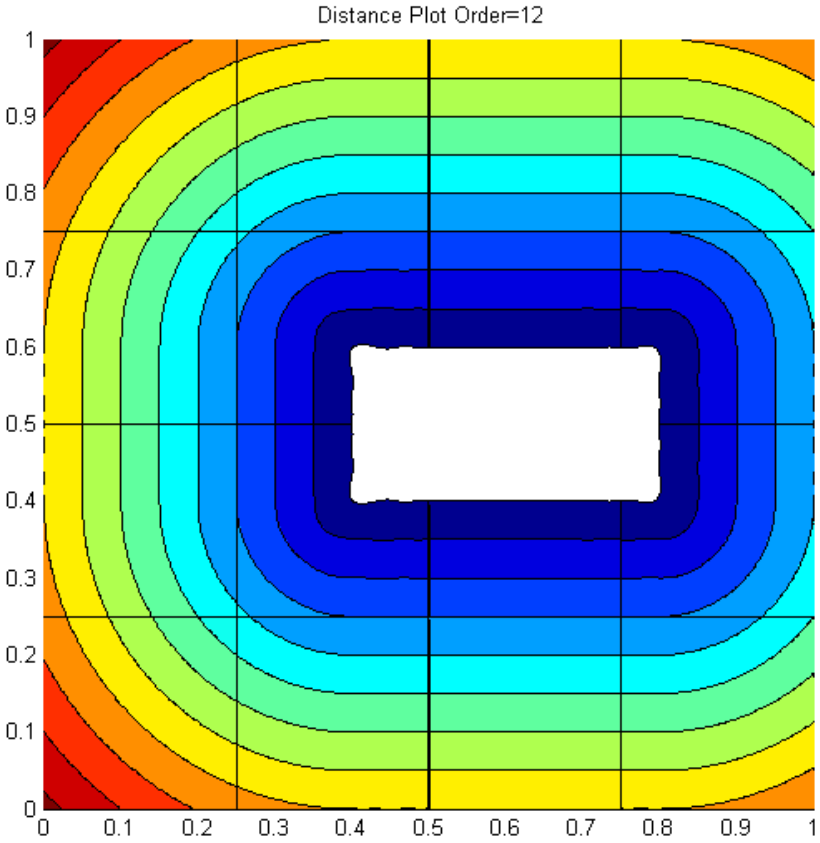
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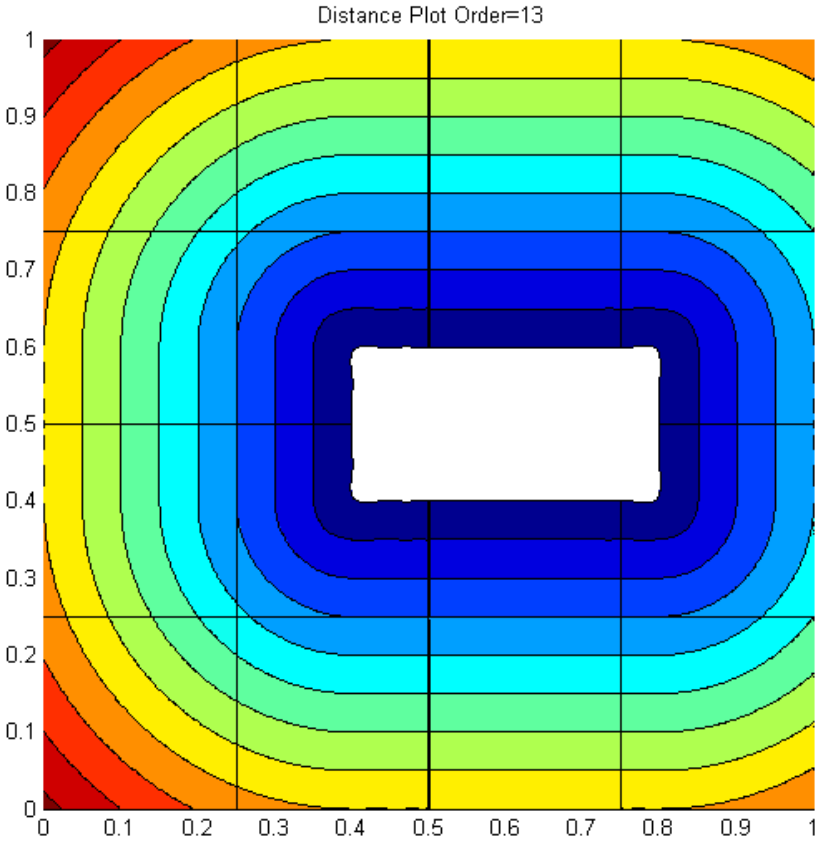
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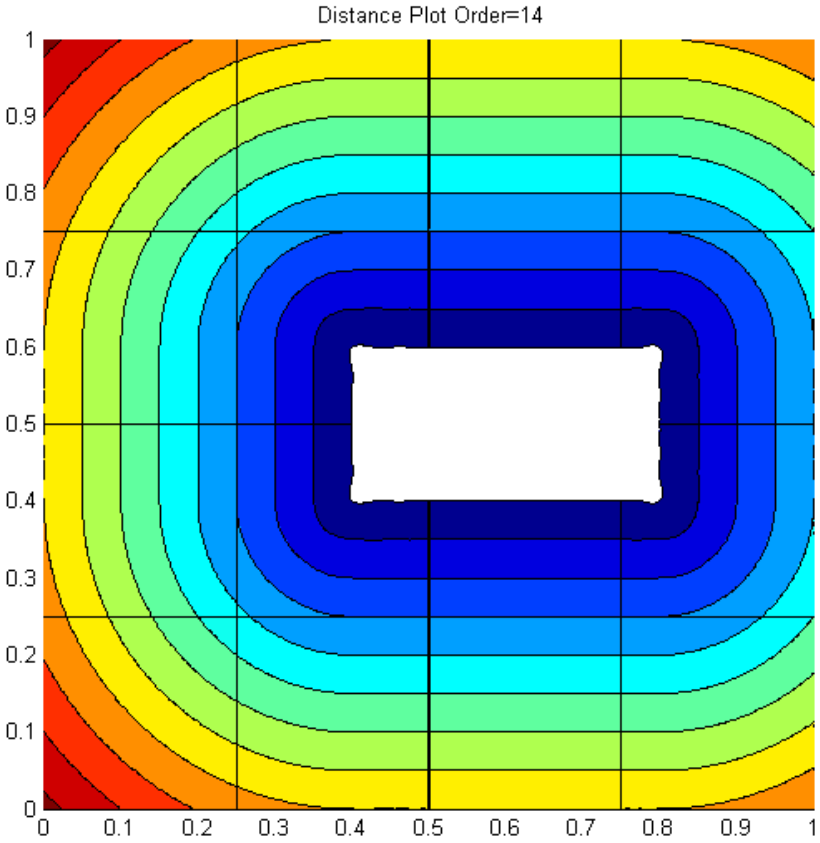
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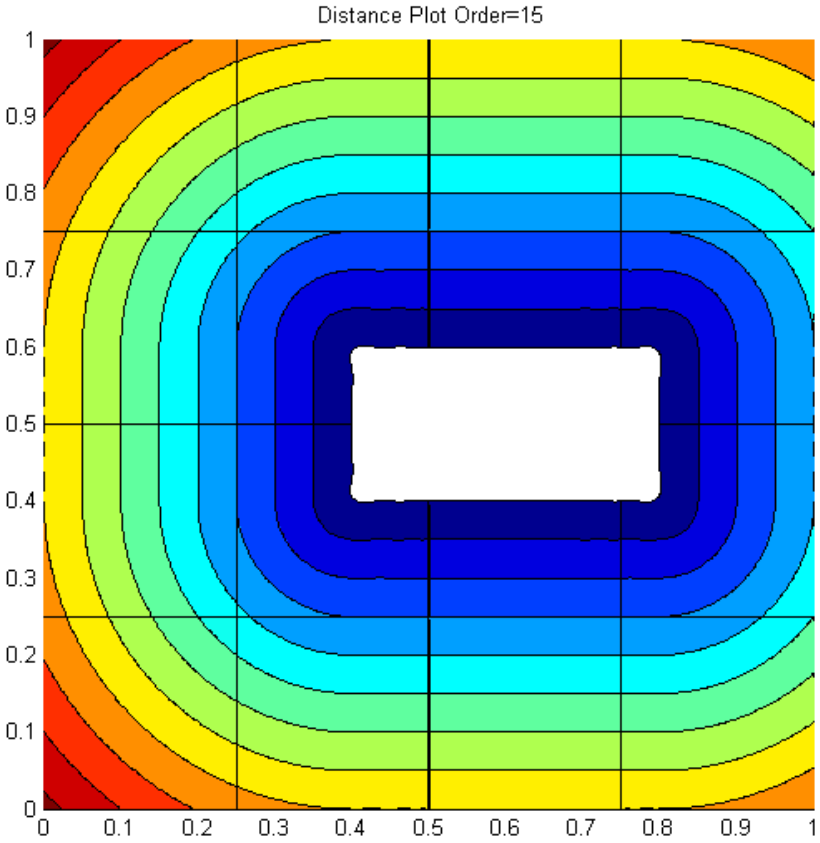
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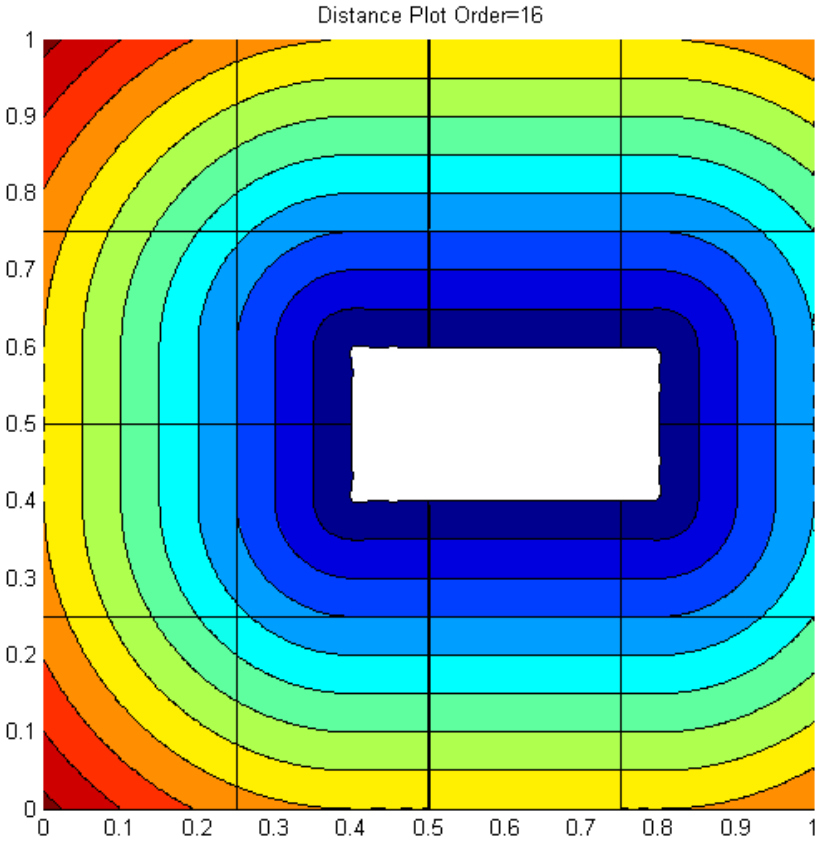
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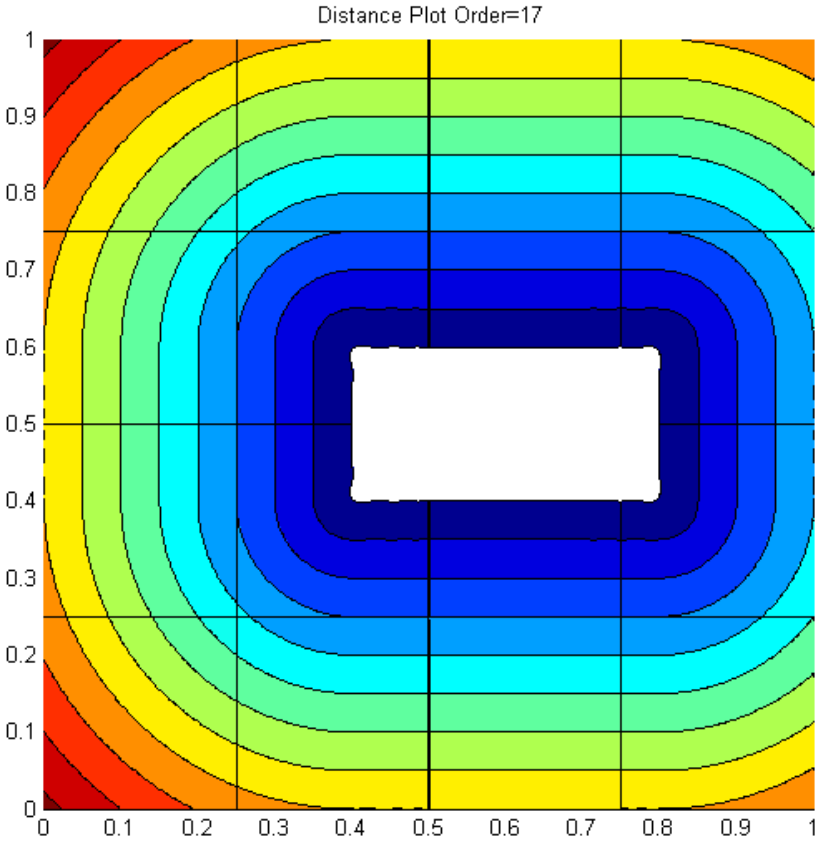
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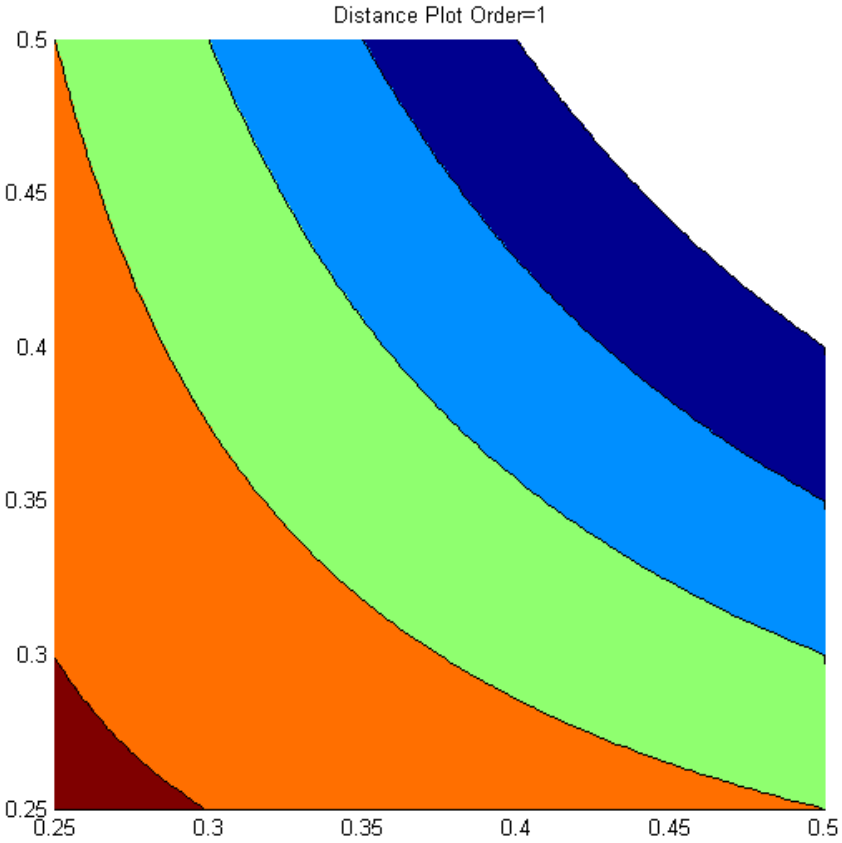
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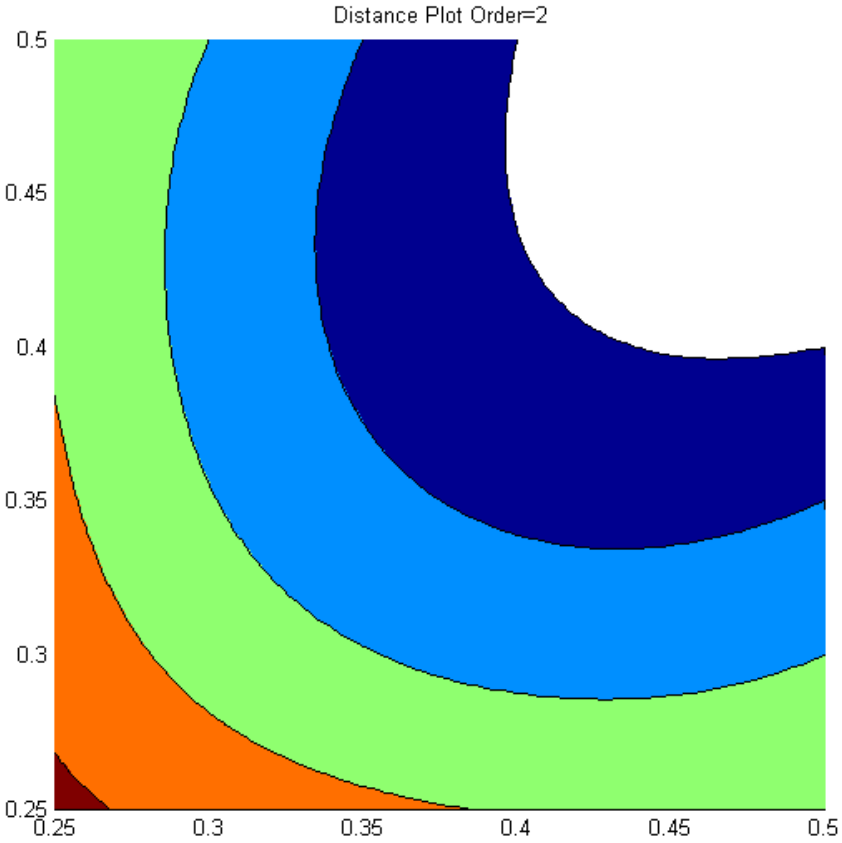
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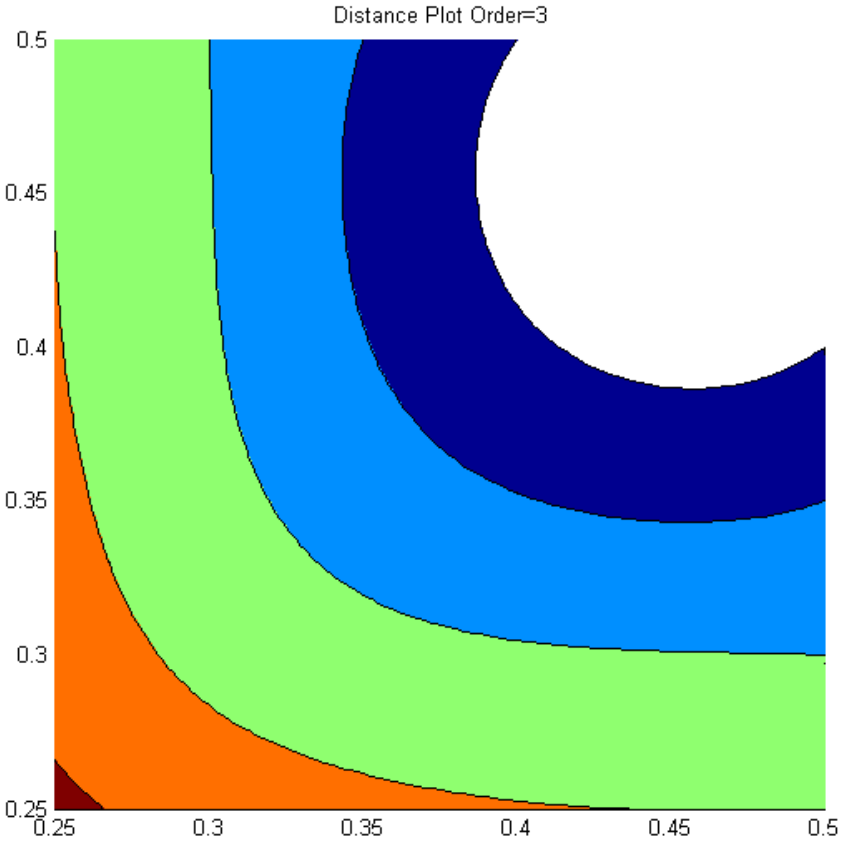
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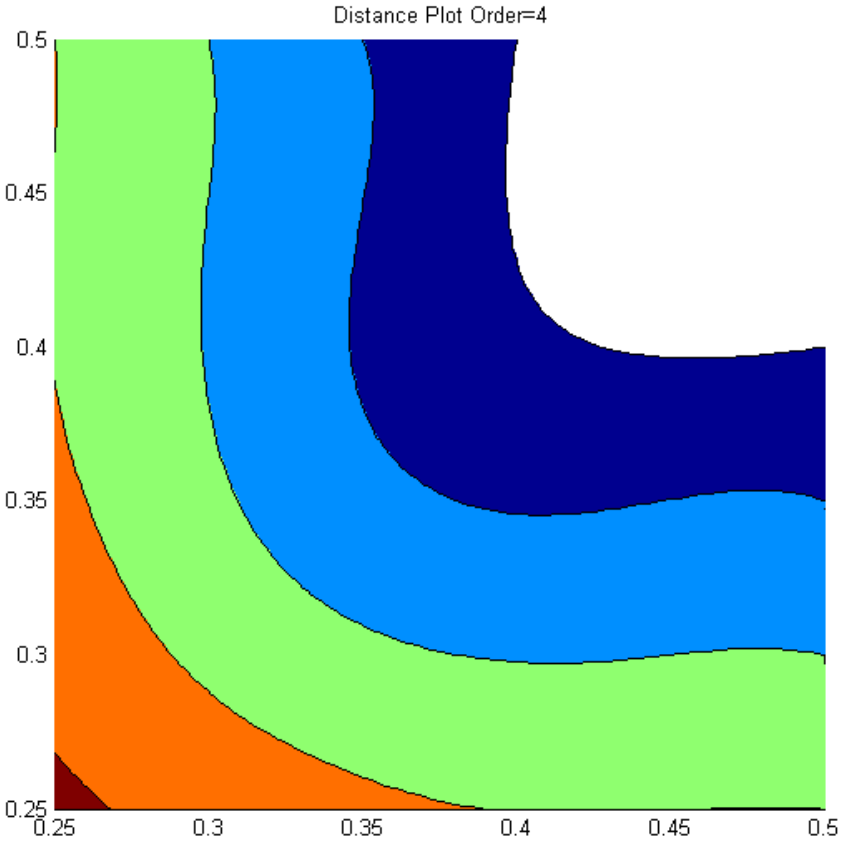
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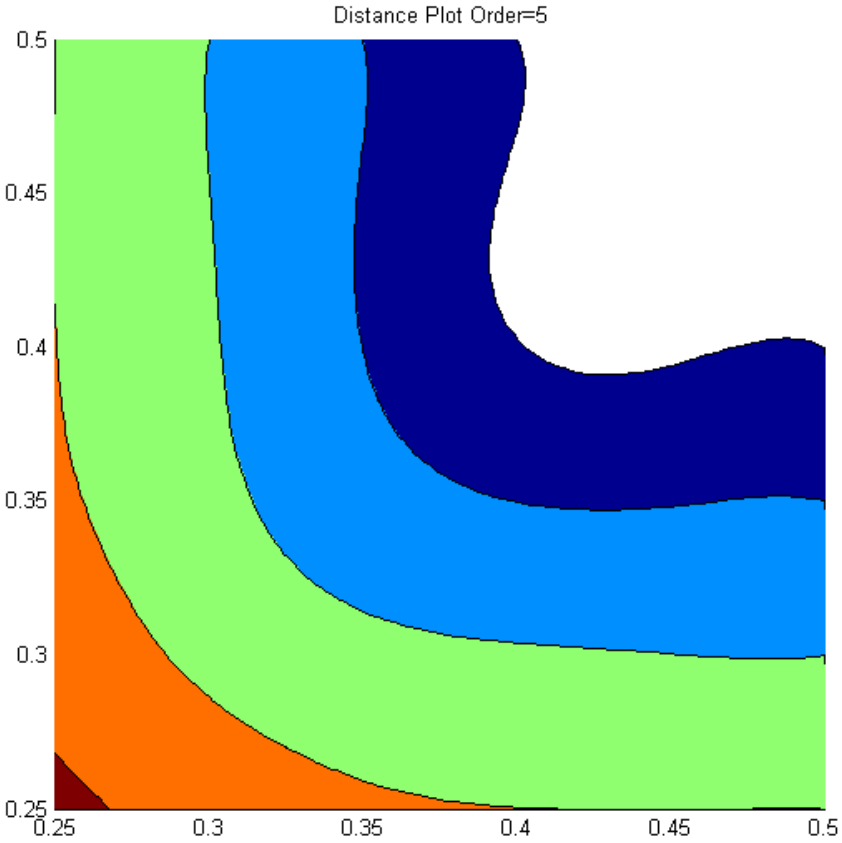
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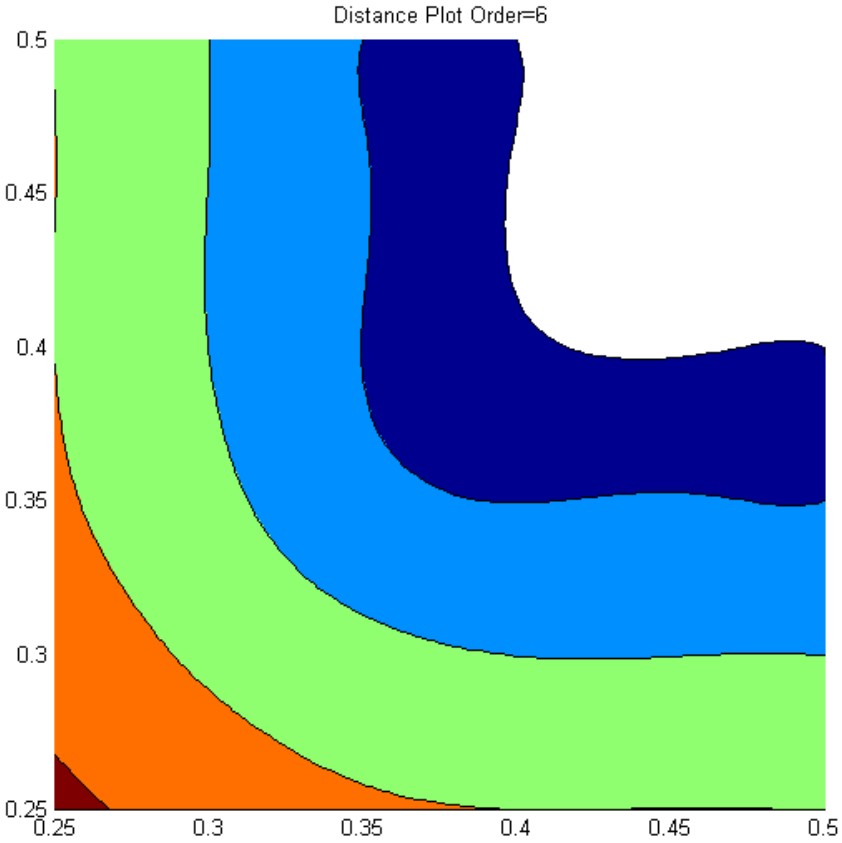
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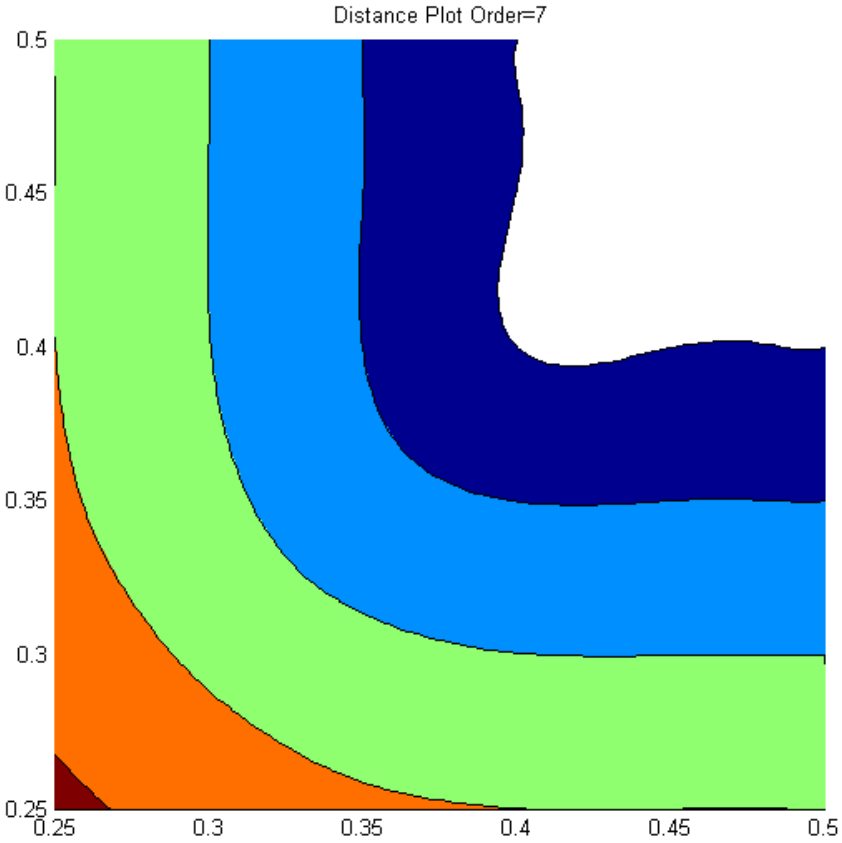
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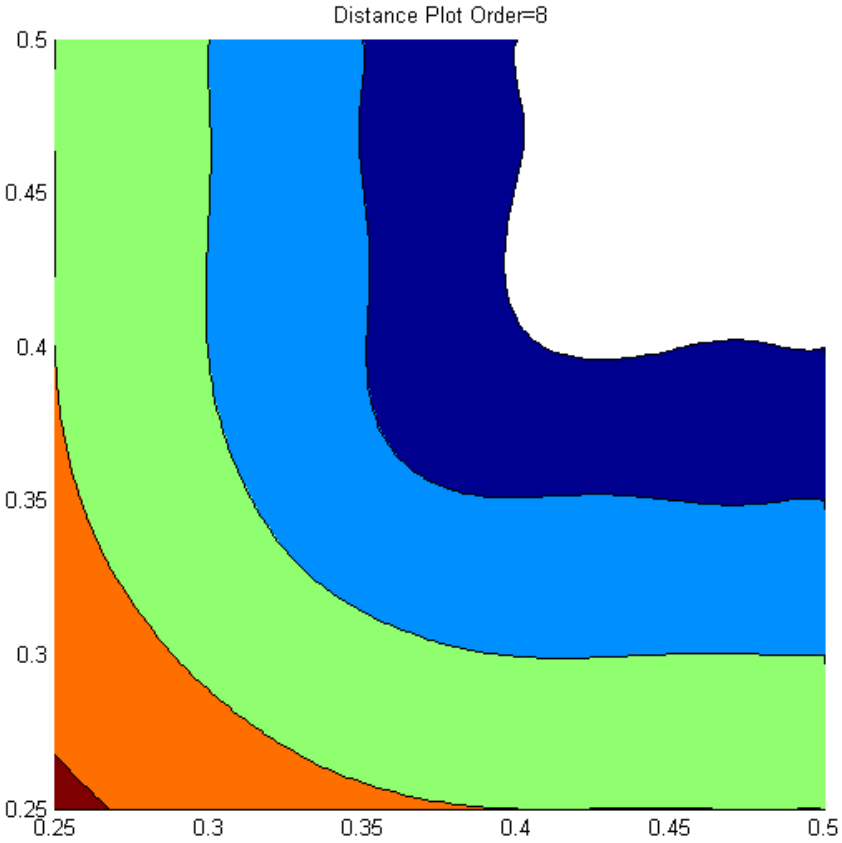
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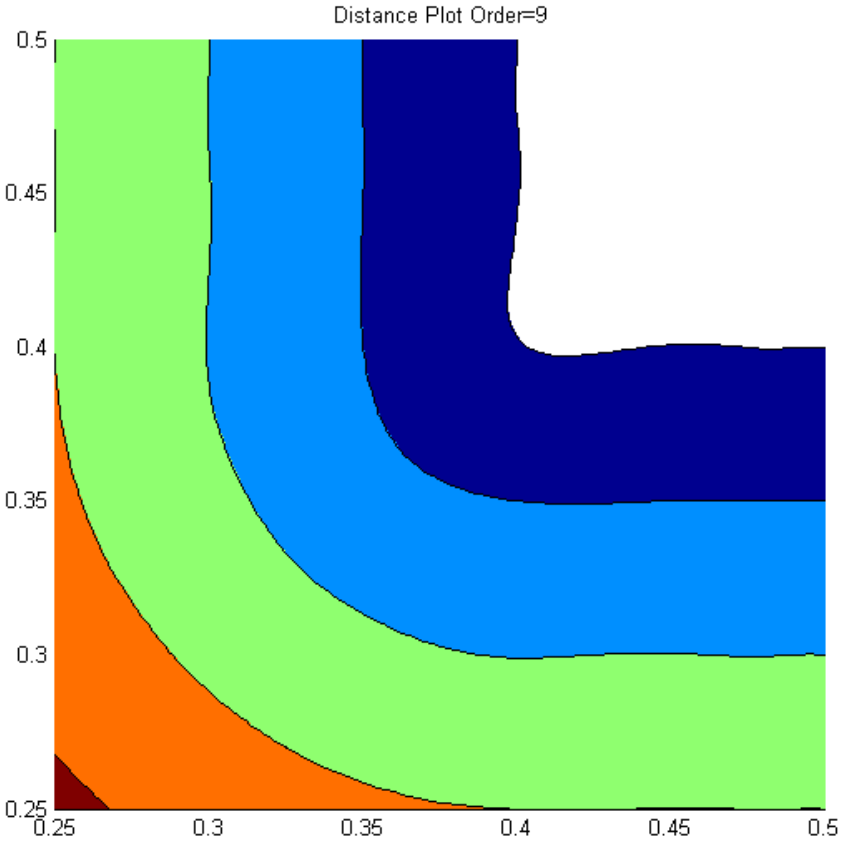
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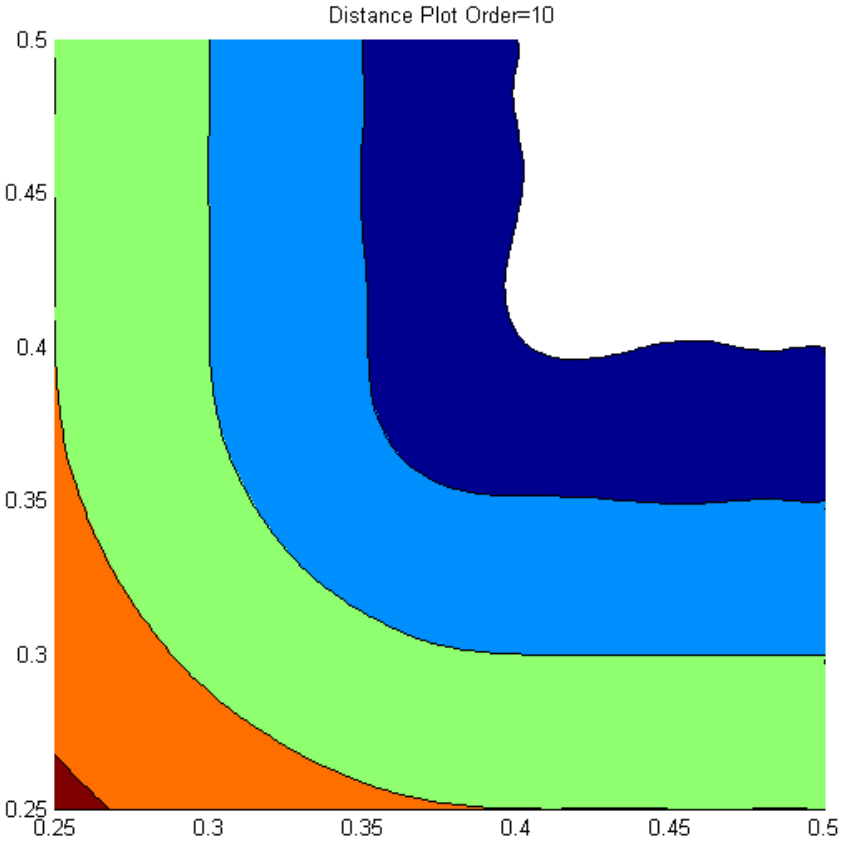
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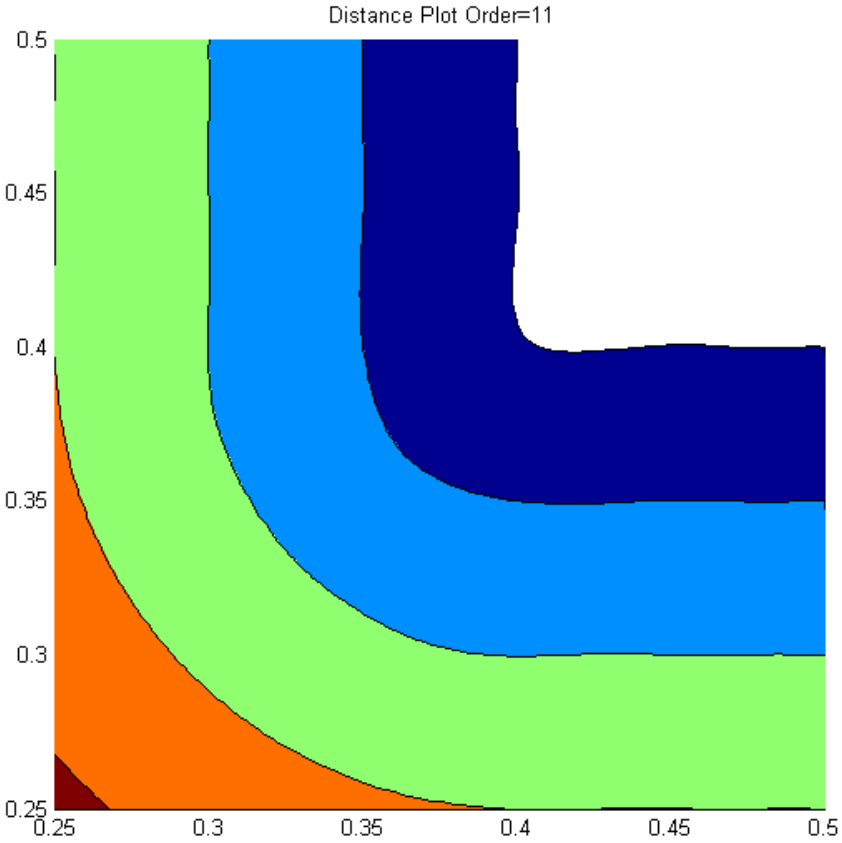
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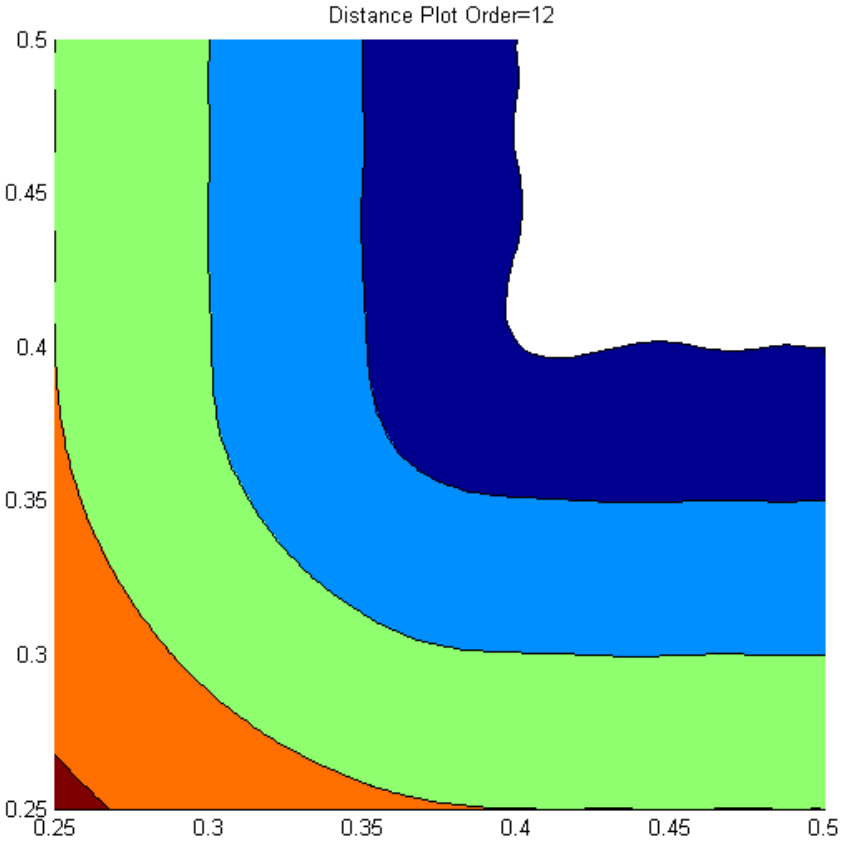
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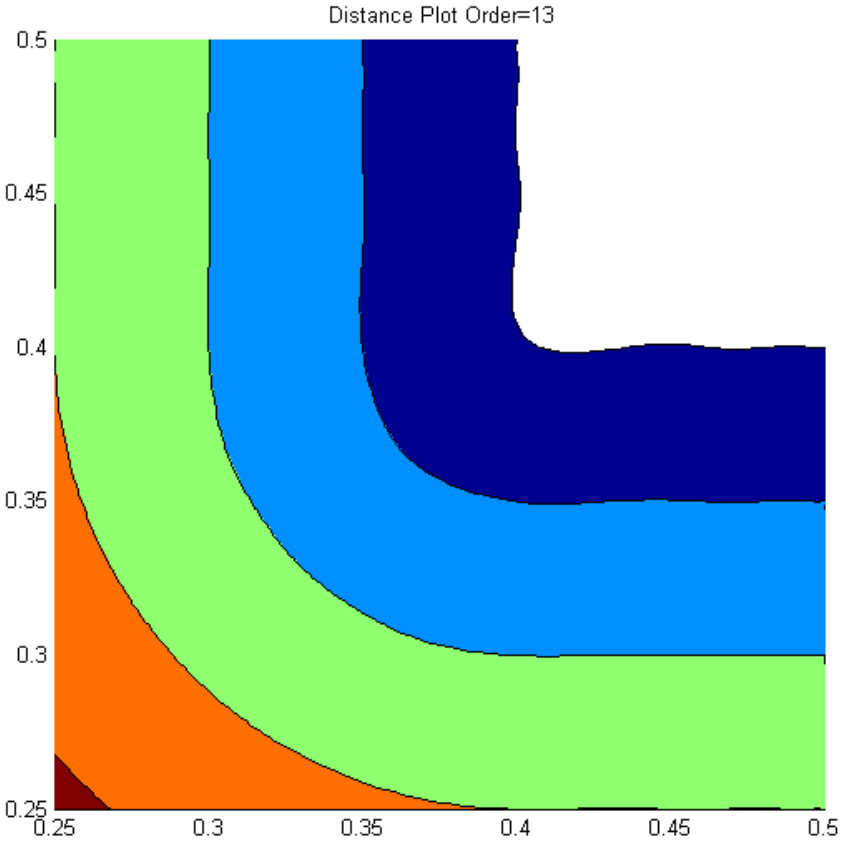
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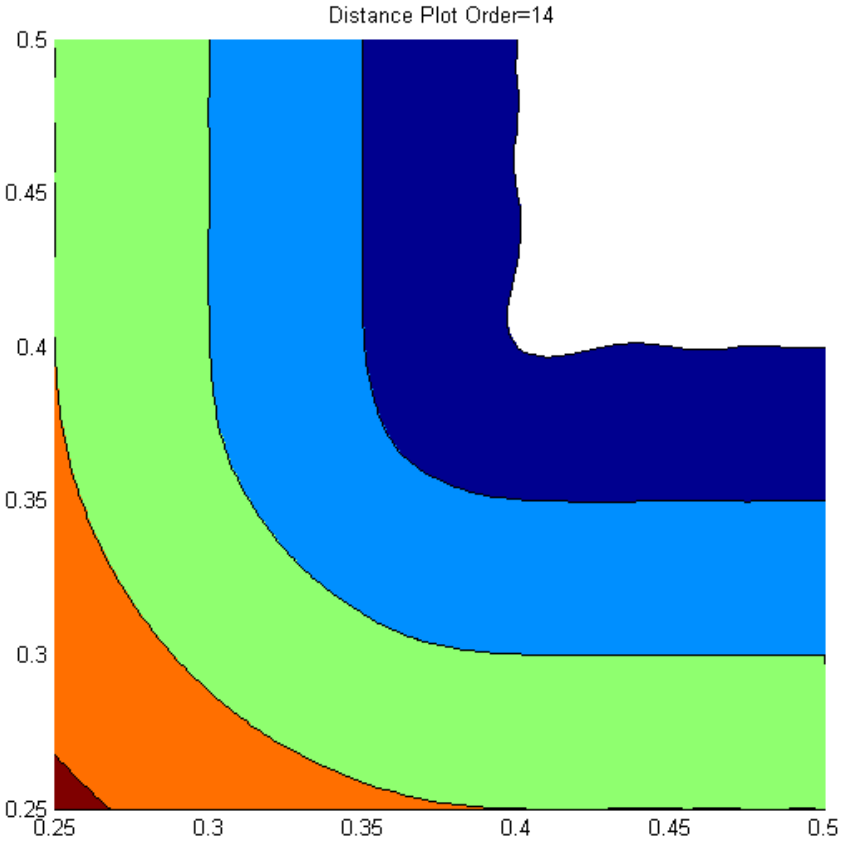
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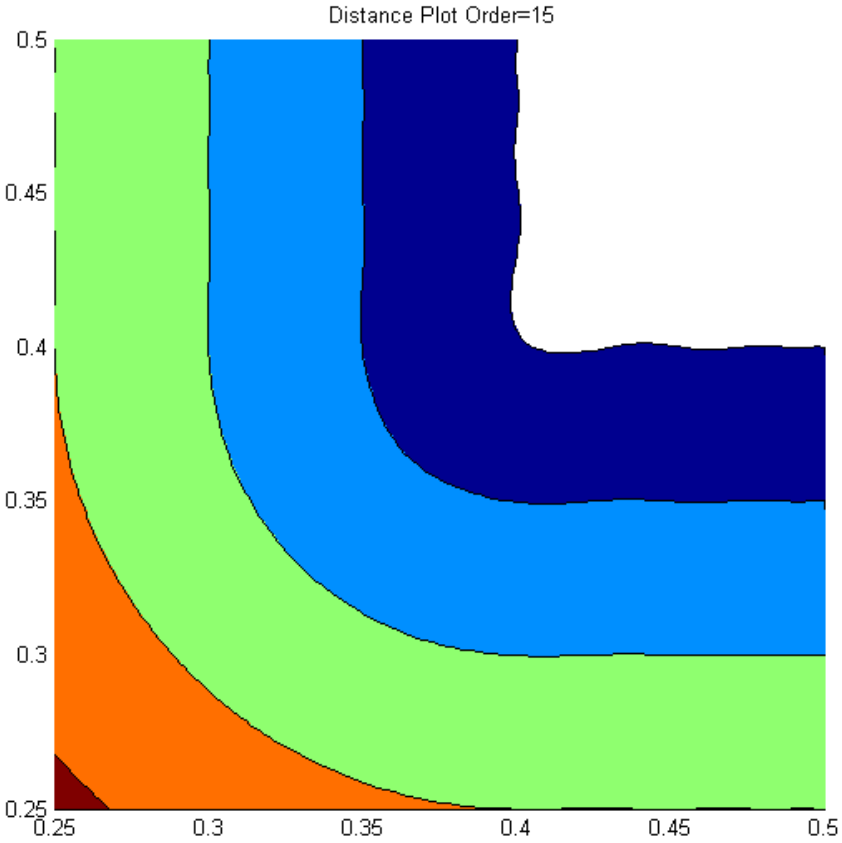
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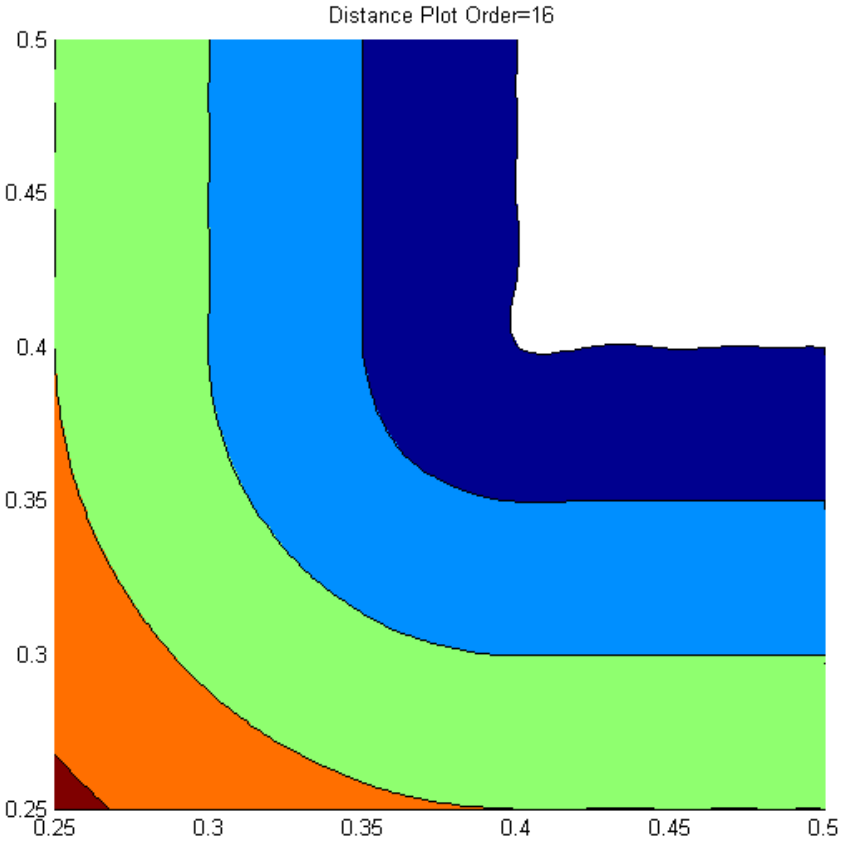
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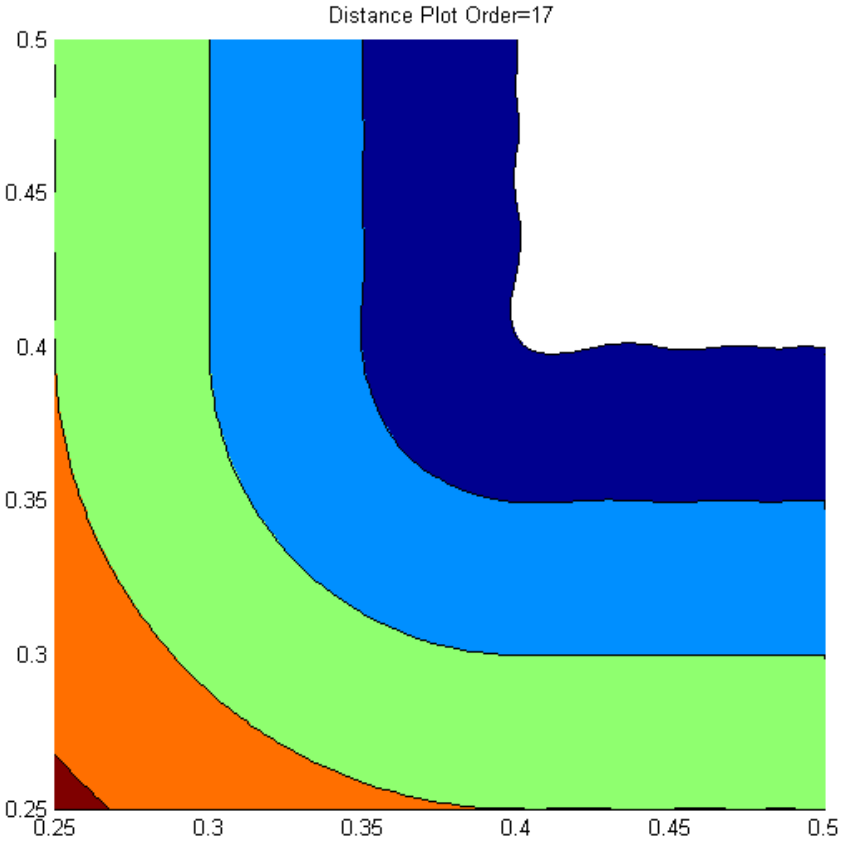
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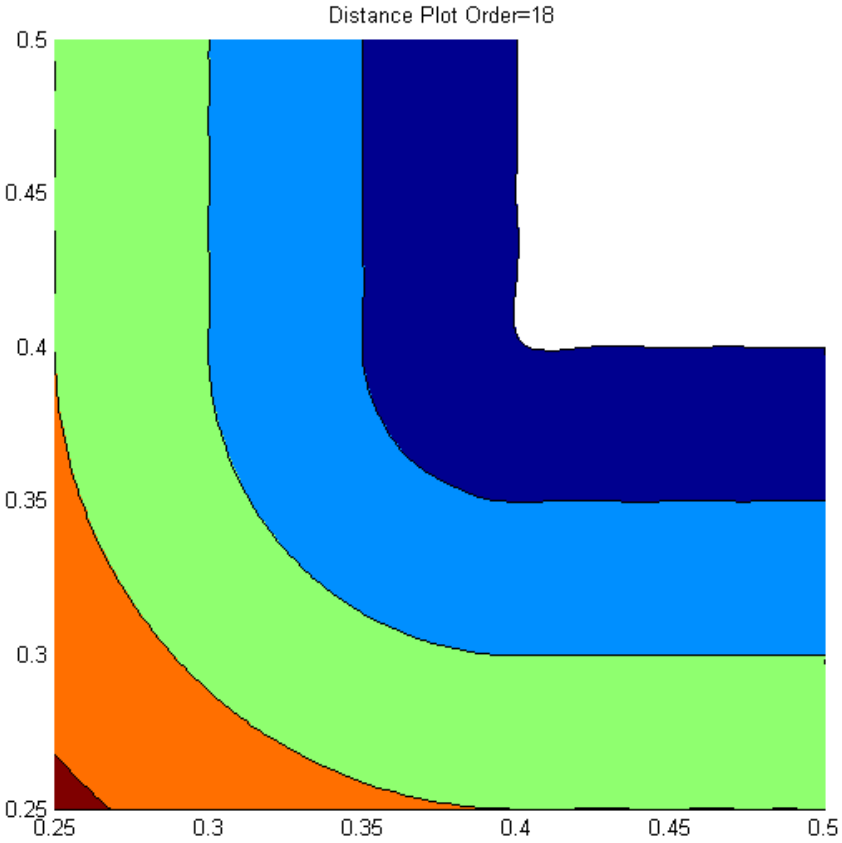
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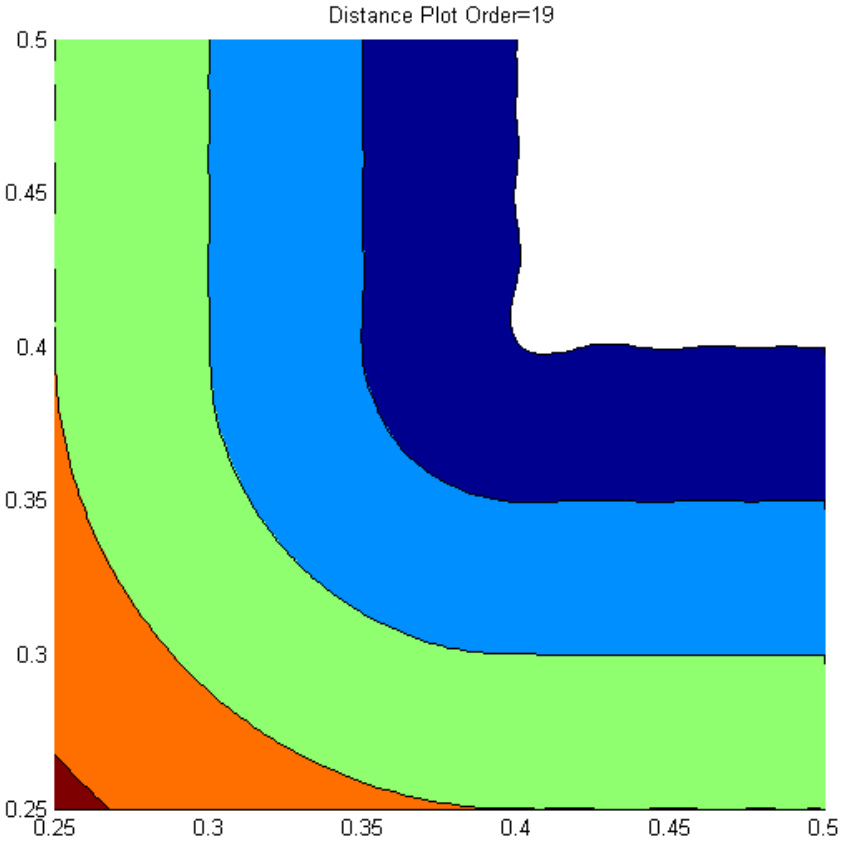
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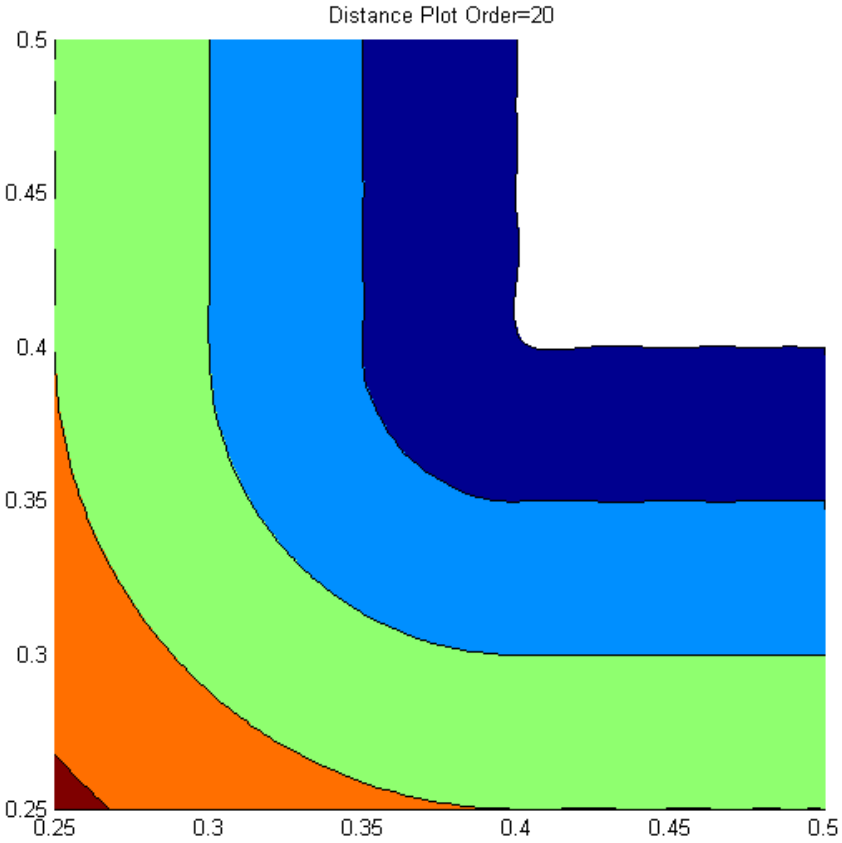
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